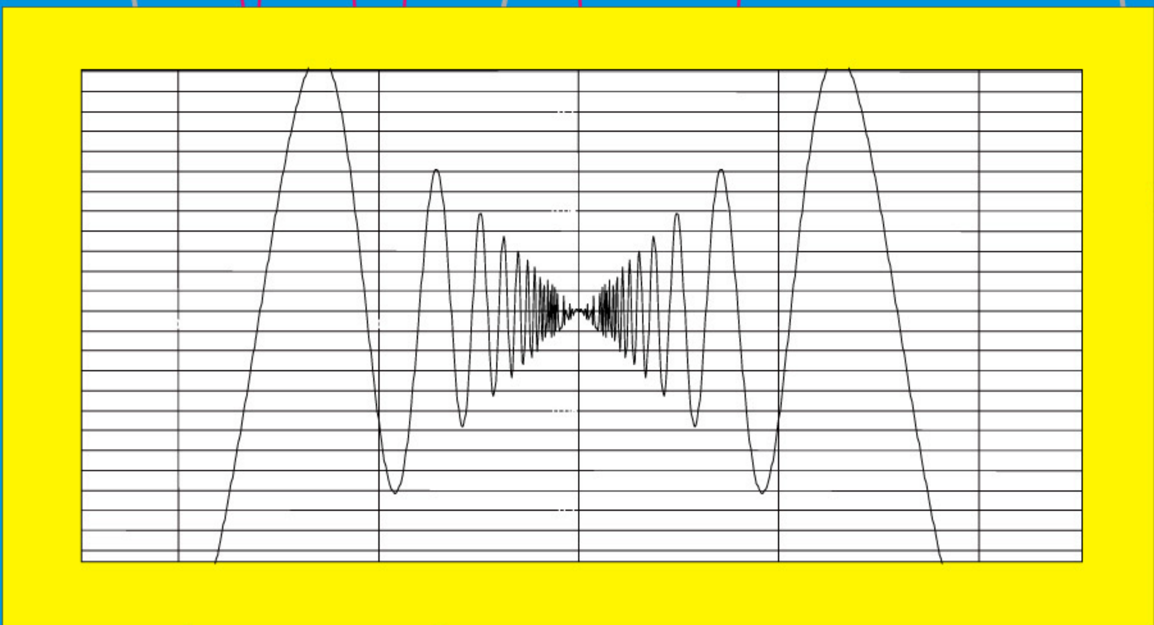
Sergiy Klymchuk

COUNTER-EXAMPLES IN CALCULUS



Foreword by John Mason

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Counter-Examples in Calculus

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Foreword

This book is a welcome and refreshing antidote to the descending spiral of instrumentality. It is offered to those students and those teachers who know that there is more to learning mathematics than completing homework mechanically. It is consistent with the view put forward by A. Watson and J. Mason in “Mathematics as a Constructive Activity: The Role of Learner-Generated Examples” (Mahwah: Erlbaum, 2005) that mathematics is a constructive activity, and that a central aspect of learning mathematics is enriching the space of examples which come to mind and to which you have access when you encounter a technical term.

An excellent way to do this is to become familiar with a wide range of examples, sometimes called ‘pathological’, but only because they are unfamiliar and even unexpected. The care and precision needed to do and to use mathematics depends upon and requires people to extend their range of ‘familiar’ examples.

One of the classic behaviours of students trying to use mathematics in another discipline is a cavalier attitude to conditions and constraints. Desperate to get a task completed, scant regard is paid to conditions which are necessary in order to apply a theorem or technique. By making the search for counter-examples an integral part of the way they expose students to mathematics, teachers can imbue all students with a more mathematical way of approaching and using mathematics.

This book provides the groundwork on which to ascend the spiral of instrumentality towards appreciation and understanding of the mathematics behind the calculus.

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Preface

This book is a supplementary resource intended to enhance the teaching and learning of a first-year university Calculus course. It can also be used in upper secondary school. It consists of carefully constructed *incorrect* mathematical statements that require students to create counter-examples to disprove them. Some of these statements are the converse of famous theorems, others are created by omitting or changing conditions of the theorems. Some are incorrect definitions and some are seemingly correct statements. Many of the statements are related to common students' misconceptions. In this book the following major topics from a typical single-variable Calculus course are explored: Functions, Limits, Continuity, Differential Calculus and Integral Calculus.

There is a well-known book on counter-examples in Calculus: "Counterexamples in Analysis" by B.*R*.Gelbaum and J.M.H.Olmsted (Holden-Day, Inc., San Francisco, 1964). It is an excellent resource for the teaching and learning of Calculus at an advanced level, but it is well beyond the scope of first-year university Calculus courses, ones that might be based on the popular "Calculus: Concepts and Contexts" by J. Stewart (Brooks/Cole, Thomson Learning, 2nd ed., 2001) for example. Compared to the above mentioned book "Counterexamples in Analysis" in this book the level is lower and the examples are easier. So these two books are not overlapping – all statements and examples are different. Unlike in "Counterexamples in Analysis", all functions used as counter-examples in this book are illustrated by their graphs, making it visually accessible and easy to understand for students.

This book is aimed at filling the niche in the activity on using counter-examples as a pedagogical strategy in teaching/learning of a first-year university Introductory Calculus course. It can be useful for:

- upper secondary school teachers and university lecturers as a teaching resource
- upper secondary school and first-year university students as a learning resource
- upper secondary school teachers for their professional development in both mathematics and mathematics education

Why Counter-Examples?

In the information age analysing given information and making a quick decision on whether it is true or false is an important ability. A counter-example is an example that shows that a given statement (conjecture, hypothesis, proposition, rule) is false. It only takes one counter-example to disprove a statement. Counter-examples play an important role in mathematics and other subjects. They are a powerful and effective tool for scientists, researchers and practitioners. They are good indicators that show that a suggested hypothesis or chosen direction of research is wrong. Before trying to prove the conjecture or hypothesis it is often worth looking for a possible counter-example. Doing so can save a lot of time and effort.

Counter-examples also provide an important means of communicating ideas in mathematics, whose entire history may be viewed as making conjectures and then either proving or disproving them by counter-examples. Here are a few well-known cases to illustrate the point:

1. For a long time mathematicians tried to find a formula for prime numbers. The numbers of the form $2^{2^n}+1$, where *n* is natural were once considered as prime numbers, until a counter-example was found. For *n* = 5 that number is composite: $2^{2^5}+1=641\times 6700417$.

2. Another conjecture about prime numbers is still waiting to be proved or disproved - Goldbach's or the Goldbach-Euler conjecture, posed by Goldbach in his letter to Euler in 1742. It looks deceptively simple at first. It states that *every even number greater than 2 is the sum of 2 prime numbers*. For example, 12 = 5 + 7, 20 = 3 + 17, and so on. A powerful computer was used in 1999 to search for counter-examples to that conjecture. No counter-examples have been found up to 4×10^{14}. In 2000 the book publishing company Faber & Faber offered a US$1 million prize to anyone who could prove or disprove that conjecture. To date (April, 2005) the prize remains unclaimed.

3. In the 19th century the great German mathematician Weierstrass constructed his famous counter-example – the first known fractal –

to the statement: *a function continuous on (a,b) cannot be non-differentiable at any point on (a,b).* Many mathematicians at that time thought that such 'monster-functions' that were continuous but not differentiable at any point were absolutely useless for practical applications. About a hundred years later Norbert Winer, the founder of cybernetics pointed out in his book "I am a mathematician" that such curves exist in nature – for example, they are trajectories of particles in Brownian motion. In recent decades such curves have been investigated in the theory of fractals – a fast growing area with many applications.

The intention of this book is to encourage teachers and students to use counter-examples in the teaching/learning of Calculus with these purposes:

- For deeper conceptual understanding
- To reduce or eliminate common misconceptions
- To advance one's mathematical thinking, that is neither algorithmic nor procedural
- To enhance generic critical thinking skills – analysing, justifying, verifying, checking, proving which can benefit students in other areas of life
- To expand the 'example set' - a number of examples of interesting functions for better communication of ideas in mathematics and in practical applications
- To make learning more active and creative

1. For deeper conceptual understanding

Many students nowadays are used to concentrating on techniques, manipulations, familiar procedures and don't pay much attention to the concepts, conditions of the theorems, properties of the functions, and to reasoning and justification.

'When students come to apply a theorem or technique, they often fail to check that the conditions for applying it are satisfied. We conjecture that this is usually because they simply do not think of it, and this is because they are not fluent in using appropriate terms, notations, properties, or do not recognise the role of such conditions' (Mason & Watson, 2001). Paying attention to the conditions of theorems helps engineering students develop the good habit of considering the extreme conditions new devices will be subjected to. Aircraft are designed to fly in storms and turbulence, not just in perfect weather! The ability to pay attention to the conditions of a sale offer is essential in everyday life. We all know the importance of reading the fine print on advertisements 'special conditions apply'.

A recent case study done in New Zealand (Klymchuk, 2005 – appendix 3) showed that the usage of counter-examples in teaching significantly improved the students' performance on test questions that required conceptual understanding.

2. To reduce or eliminate misconceptions

Over recent years, partly due to extensive usage of modern technology, the proof component of the traditional approach in teaching Calculus (definition-theorem-proof-example-application) has almost disappeared. Students are used to relying on technology and sometimes lack logical thinking and conceptual understanding. Sometimes Calculus courses are taught in such a way that special cases are avoided and students are exposed only to 'nice' functions and 'good' examples, especially at school level. This approach can create many misconceptions that can be explained by Tall's generic extension principle: 'If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts.' (Tall, 1991).

In this book many wrong statements are related to students' common misconceptions. There is a difference between students' misconceptions in basic algebra and in Calculus. There are no textbooks where 'properties' like $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ can be found, and nobody teaches such 'rules' either. Some introductory Calculus textbooks on the other hand, especially those at school level, contain incorrect statements. For example: "*If the graph of a function is a continuous and smooth curve* (*no sharp corners*) *on (a,b) then the function is differentiable on (a,b)*", and "*a tangent line to a curve is a line that just touches the curve at one point and does not cross it there*". Some students actually *learn* Calculus this way. Practice in creating counter-examples can help students reduce or eliminate such misconceptions before they become second nature.

3. To advance mathematical thinking

Creating examples and counter-examples is neither algorithmic nor procedural and requires advanced mathematical thinking which is not often taught at school. 'Coming up with examples requires different cognitive skills from carrying out algorithms – one needs to look at mathematical objects in terms of their properties. To be asked for an example can be disconcerting. Students have no pre-learned algorithms to show the "correct way" (Selden & Selden, 1998). Practice in constructing their own examples and counter-examples can help students enhance their creativity and advance their mathematical thinking.

4. To enhance generic critical thinking skills

Creating counter-examples to wrong statements has a big advantage over constructing examples of functions satisfying certain conditions, because counter-examples deal with disproving, justification, argumentation, reasoning and critical thinking, which are the essence of mathematical thinking. These skills will benefit students not only in their university study but also in other areas of life.

5. To expand the 'example set'

After creating or being exposed to many functions with interesting properties students will expand their 'example set', allowing them to better communicate their ideas in mathematics and in practical applications. While creating counter-examples students learn a lot about the behaviour of functions and can later apply their knowledge to solving real life problems.

For example:

a) the counter-examples to statement 2 (Limits) and statement 32 (Differential Calculus) from the book are the functions $f(x)=\frac{\sin x}{x}$ and $f(x)=\begin{cases} x^2\left|\cos\frac{\pi}{x}\right|, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$ respectively, which are used for modelling vibration processes in mechanical engineering;

b) the counter-example to statement 12 (Integral Calculus) is the Fresnel function $F(x) = \int_0^x \sin(\frac{\pi t^2}{2})dt$ which apart from being important in optics has recently been applied to motorway design.

As Henry Pollak from Bell Laboratories, USA pointed out "the society provides time for mathematics to be taught in schools, colleges and universities not because mathematics is beautiful, which it is, or because it provides a great training for the mind, but because it is so useful".

6. To make learning more active and creative

Experience of my colleagues and my own teaching experience shows that the usage of counter-examples as a pedagogical strategy in lectures and assignments can create a discovery learning environment and make learning more active. A recent international study involving more than 600 students from 10 universities in different countries (Gruenwald & Klymchuk, 2003 – appendix 2) showed that the vast majority of the participating students (92%) found the use of counter-examples to be very effective. They reported it helped them to understand concepts better, prevent mistakes, develop logical and critical thinking, and that they were more actively involved in lectures. Many commented that creating a variety of counter-examples enhanced their critical thinking skills in general, skills useful in other areas of life that have nothing to do with mathematics.

There are different ways of using counter-examples in teaching:

- giving the students a mixture of correct and incorrect statements
- making a deliberate mistake in the lecture
- asking the students to spot an error on a certain page of their textbook
- giving the students bonus marks towards their final grade for providing excellent counter-examples to hard questions during the lecture.

It can also be a part of assessment.

The Structure of the Book

The first part of the book contains incorrect statements from the five major topics found in Introductory Calculus courses: Functions, Limits, Continuity, Differential Calculus and Integral Calculus. The statements from each topic are arranged in order of increasing difficulty. Some statements, especially those in the beginning of each topic, are related to students' regular misunderstandings. In the more challenging cases statements often appear to be correct, and students will be hard-pressed to find counter-examples to them. I believe all readers will find interesting and surprising examples in the book.

The second part of the book contains suggested solutions to all the statements. It is anticipated that the readers will construct their own counter-examples to the statements. Some solutions are followed by comments written mainly for the students.

The book contains three appendices - an example from teaching practice and two papers on mathematics education related to the content of this book. The first paper deals with the students' attitudes towards using counter-examples in teaching/learning situations. The second paper is about students' performance after working with counter-examples. Both papers are reproduced here with kind permission from the editor of The New Zealand Mathematics Magazine.

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Statements

1. Functions

1. The tangent to a curve at a point is the line which touches the curve at that point but does not cross it there.

2. The tangent line to a curve at a point cannot touch the curve at infinitely many other points.

3. A quadratic function of x is one in which the highest power of x is two.

4. If both functions $y = f(x)$ and $y = g(x)$ are continuous and monotone on R then their sum $f(x) + g(x)$ is also monotone on R.

5. If both functions $y = f(x)$ and $y = g(x)$ are not monotone on R then their sum $f(x) + g(x)$ is not monotone on R.

6. If a function $y = f(x)$ is continuous and decreasing for all positive x and $f(1)$ is positive then the function has exactly one root.

7. If a function $y = f(x)$ has an inverse function $\boldsymbol{x = f^{-1}(y)}$ on (a,b) then the function $f(x)$ is either increasing or decreasing on (a,b).

8. A function $y = f(x)$ is bounded on R if for any $\boldsymbol{x \in R}$ there is $M > 0$ such that $\boldsymbol{|f(x)| \le M}$.

9. If $g(a) = 0$ then the function $\boldsymbol{F(x) = \dfrac{f(x)}{g(x)}}$ has a vertical asymptote at the point $x = a$.

10. If $g(a) = 0$ then the *rational* function $\boldsymbol{R(x) = \dfrac{f(x)}{g(x)}}$ (both $f(x)$ and $g(x)$ are polynomials) has a vertical asymptote at the point $x = a$.

11. If a function $y = f(x)$ is unbounded and non-negative for all real x then it cannot have roots x_n such that $x_n \to \infty$ as $n \to \infty$.

12. A function $y = f(x)$ defined on $[a,b]$ such that its graph doesn't contain any pieces of a horizontal straight line cannot take its extreme value infinitely many times on $[a,b]$.

13. If a function $y = f(x)$ is continuous and increasing at the point $x = a$ then there is a neighbourhood $(x-\delta, x+\delta), \delta > 0$ where the function is also increasing.

14. If a function is not monotone then it doesn't have an inverse function.

15. If a function is not monotone on (a,b) then its square cannot be monotone on (a,b).

2. Limits

1. If $f(x) < g(x)$ for all $x > 0$ and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist then $\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x)$.

2. The following definitions of a non-vertical asymptote are equivalent:

   a) The straight line $y = mx + c$ is called a non-vertical asymptote to a curve $f(x)$ as x tends to infinity if $\lim_{x\to\infty}(f(x) - (mx + c)) = 0$.

   b) A straight line is called a non-vertical asymptote to a curve as x tends to infinity if the curve gets closer and closer (as close as we like) to the straight line as x tends to infinity without touching or crossing it.

3. The tangent line to a curve at a certain point that touches the curve at infinitely many other points cannot be a non-vertical asymptote to this curve.

4. The following definitions of a vertical asymptote are equivalent:

   a) The straight line $x = a$ is called a vertical asymptote for a function $y = f(x)$ if $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

   b) The straight line $x = a$ is called a vertical asymptote for the function $y = f(x)$ if there are infinitely many values of $f(x)$ that can be made arbitrarily large or arbitrarily small as x gets closer to a from either side of a.

5. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ doesn't exist because of oscillation of $g(x)$ near $x = a$ then $\lim_{x\to a}(f(x) \times g(x))$ doesn't exist.

6. If a function $y = f(x)$ is not bounded in any neighbourhood of the point $x = a$ then either $\lim_{x\to a^+}|f(x)| = \infty$ or $\lim_{x\to a^-}|f(x)| = \infty$.

7. If a function $y = f(x)$ is continuous for all real x and $\lim_{n\to\infty} f(n) = A$ then $\lim_{x\to\infty} f(x) = A$.

3. Continuity

1. If the absolute value of the function $y = f(x)$ is continuous on (a,b) then the function is also continuous on (a,b).

2. If both functions $y = f(x)$ and $y = g(x)$ are discontinuous at $x = a$ then $f(x) + g(x)$ is also discontinuous at $x = a$.

3. If both functions $y = f(x)$ and $y = g(x)$ are discontinuous at $x = a$ then $\boldsymbol{f(x)\times g(x)}$ is also discontinuous at $x = a$.

4. A function always has a local maximum between any two local minima.

5. For a continuous function there is always a local maximum between any two local minima.

6. If a function is defined in a certain neighbourhood of point $x = a$ including the point itself and is increasing on the left from $x = a$ and decreasing on the right from $x = a$, then there is a local maximum at $x = a$.

7. If a function is defined on $[a,b]$ and continuous on (a,b) then it takes its extreme values on $[a,b]$.

8. Every continuous and bounded function on $(-\infty,\infty)$ takes on its extreme values.

9. If a function $y = f(x)$ is continuous on $[a,b]$, the tangent line exists at all points on its graph and $f(a) = f(b)$ then there is a point c in (a,b) such that the tangent line at the point $(c,f(c))$ is horizontal.

10. If on the closed interval $[a,b]$ a function is:
    a. bounded;
    b. takes its maximum and minimum values;
    c. takes all its values between the maximum and minimum vales;

    then this function is continuous on $[a,b]$.

11. If on the closed interval $[a,b]$ a function is:

    a. bounded;

    b. takes its maximum and minimum values;

    c. takes all its values between the maximum and minimum values;

    then this function is continuous at one or more points or subintervals on $[a,b]$.

12. If a function is continuous on $[a,b]$ then it cannot take its absolute maximum or minimum value infinitely many times.

13. If a function $y = f(x)$ is defined on $[a,b]$ and $f(a) \times f(b) < 0$ then there is some point $c \in (a,b)$ such that $f(c) = 0$.

14. If a function $y = f(x)$ is defined on $[a,b]$ and continuous on (a,b) then for any $N \in (f(a), f(b))$ there is some point $c \in (a,b)$ such that $f(c) = N$.

15. If a function is discontinuous at every point in its domain then the square and the absolute value of this function cannot be continuous.

16. A function cannot be continuous at only one point in its domain and discontinuous everywhere else.

17. A sequence of continuous functions on $[a,b]$ always converges to a continuous function on $[a,b]$.

4. Differential Calculus

1. If both functions $y = f(x)$ and $y = g(x)$ are differentiable and $f(x) > g(x)$ on the interval (a,b) then $f'(x) > g'(x)$ on (a,b).

2. If a non-linear function is differentiable and monotone on $(0,\infty)$ then its derivative is also monotone on $(0,\infty)$.

3. If a function is continuous at a point then it is differentiable at that point.

4. If a function is continuous on R and the tangent line exists at any point on its graph then the function is differentiable at any point on R.

5. If a function is continuous on the interval (a,b) and its graph is a *smooth* curve (no sharp corners) on that interval then the function is differentiable at any point on (a,b).

6. If the derivative of a function is zero at a point then the function is neither increasing nor decreasing at this point.

7. If a function is differentiable and decreasing on (a,b) then its gradient is negative on (a,b).

8. If a function is continuous and decreasing on (a,b) then its gradient is non-positive on (a,b).

9. If a function has a positive derivative at any point in its domain then the function is increasing everywhere in its domain.

10. If a function $y = f(x)$ is defined on $[a,b]$ and has a local maximum at the point $c \in (a,b)$ then in a sufficiently small neighbourhood of the point $x = c$ the function is increasing on the left and decreasing on the right from $x = c$.

11. If a function $y = f(x)$ is differentiable for all real x and $f(0) = f'(0) = 0$ then $f(x) = 0$ for all real x.

12. If a function $y = f(x)$ is differentiable on the interval (a,b) and takes both positive and negative values on it then its absolute value $|f(x)|$ is not differentiable at the point(s) where $f(x) = 0$, e.g. $|f(x)| = |x|$ or $|f(x)| = |\sin x|$.

13. If both functions $y = f(x)$ and $y = g(x)$ are differentiable on the interval (a,b) and intersect somewhere on (a,b) then the function $\max\{f(x), g(x)\}$ is not differentiable at the point(s) where $f(x) = g(x)$.

14. If a function is twice differentiable at a local maximum (minimum) point then its second derivative is negative (positive) at that point.

15. If both functions $y = f(x)$ and $y = g(x)$ are non-differentiable at $x = a$ then $f(x) + g(x)$ is also not differentiable at $x = a$.

16. If a function $y = f(x)$ is differentiable and a function $y = g(x)$ is not differentiable at $x = a$ then $f(x) \times g(x)$ is not differentiable at $x = a$.

17. If both functions $y = f(x)$ and $y = g(x)$ are not differentiable at $x = a$ then $f(x) \times g(x)$ is also not differentiable at $x = a$.

18. If a function $y = g(x)$ is differentiable at $x = a$ and a function $y = f(x)$ is not differentiable at $g(a)$ then the function $F(x) = f(g(x))$ is not differentiable at $x = a$.

19. If a function $y = g(x)$ is not differentiable at $x = a$ and a function $y = f(x)$ is differentiable at $g(a)$ then the function $F(x) = f(g(x))$ is not differentiable at $x = a$.

20. If a function $y = g(x)$ is not differentiable at $x = a$ and a function $y = f(x)$ is not differentiable at $g(a)$ then the function $F(x) = f(g(x))$ is not differentiable at $x = a$.

21. If a function $y = f(x)$ is defined on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.

22. If a function is twice-differentiable in a certain neighbourhood of the point $x = a$ and its second derivative is zero at that point then the point $(a,f(a))$ is a point of inflection for the graph of the function.

23. If a function $y = f(x)$ is differentiable at the point $x = a$ and the point $(a,f(a))$ is a point of inflection on the function's graph then the second derivative is zero at that point.

24. If both functions $y = f(x)$ and $y = g(x)$ are differentiable on R then to evaluate the limit $\lim_{x\to\infty}\frac{f(x)}{g(x)}$ in the indeterminate form of type $\left[\frac{\infty}{\infty}\right]$ we can use the following rule: $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$.

25. If a function $y = f(x)$ is differentiable on (a,b) and $\lim_{x\to a^+} f'(x) = \infty$ then $\lim_{x\to a^+} f(x) = \infty$.

26. If a function $y = f(x)$ is differentiable on $(0,\infty)$ and $\lim_{x\to\infty} f(x)$ exists then $\lim_{x\to\infty} f'(x)$ also exists.

27. If a function $y = f(x)$ is differentiable and bounded on $(0,\infty)$ and $\lim_{x\to\infty} f'(x)$ exists then $\lim_{x\to\infty} f(x)$ also exists.

28. If a function $y = f(x)$ is differentiable at the point $x = a$ then its derivative is continuous at $x = a$.

29. If the derivative of a function $y = f(x)$ is positive at the point $x = a$ then there is a neighbourhood about $x = a$ (no matter how small) where the function is increasing.

30. If a function $y = f(x)$ is continuous on (a,b) and has a local maximum at the point $c \in (a,b)$ then in a sufficiently small neighbourhood of the point $x = c$ the function is increasing on the left and decreasing on the right from $x = c$.

31. If a function $y = f(x)$ is differentiable at the point $x = a$ then there is a certain neighbourhood of the point $x = a$ where the derivative of the function $y = f(x)$ is bounded.

32. If a function $y = f(x)$ at any neighbourhood of the point $x = a$ has points where $f'(x)$ doesn't exist then $f'(a)$ doesn't exist.

33. A function cannot be differentiable only at one point in its domain and non-differentiable everywhere else in its domain.

34. A continuous function cannot be non-differentiable at every point in its domain.

5. Integral Calculus

1. If the function $y = f(x)$ is an antiderivative of a function $y = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$.

2. If a function $y = f(x)$ is continuous on $[a,b]$ then the area enclosed by the graph of $y = f(x)$, OX, $x = a$ and $x = b$ numerically equals $\int_a^b f(x)dx$.

3. If $\int_a^b f(x)dx \geq 0$ then $f(x) \geq 0$ for all $x \in [a,b]$.

4. If $y = f(x)$ is a continuous function and k is any constant then:

   $\int kf(x)dx = k\int f(x)dx$.

5. A plane figure of an infinite area rotated about an axis always produces a solid of revolution of infinite volume.

6. If a function $y = f(x)$ is defined for any $x \in [a,b]$ and $\int_a^b |f(x)|dx$ exists then $\int_a^b f(x)dx$ exists.

7. If neither of the integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist then the integral $\int_a^b (f(x) + g(x))dx$ doesn't exist.

8. If $\lim_{x\to\infty} f(x) = 0$ then $\int_a^\infty f(x)dx$ converges.

9. If the integral $\int_a^\infty f(x)dx$ diverges then the function $y = f(x)$ is not bounded.

10. If a function $y = f(x)$ is continuous and non-negative for all real x and $\sum_{n=1}^{\infty} f(n)$ is finite then $\int_{1}^{\infty} f(x)dx$ converges.

11. If both integrals $\int_{a}^{\infty} f(x)dx$ and $\int_{a}^{\infty} g(x)dx$ diverge then the integral $\int_{a}^{\infty} (f(x)+g(x))dx$ also diverges.

12. If a function $y = f(x)$ is continuous and $\int_{a}^{\infty} f(x)dx$ converges then $\lim_{x\to\infty} f(x) = 0.$

13. If a function $y = f(x)$ is continuous and non-negative and $\int_{a}^{\infty} f(x)dx$ converges then $\lim_{x\to\infty} f(x) = 0$.

14. If a function $y = f(x)$ is positive and not bounded for all real x then the integral $\int_{a}^{\infty} f(x)dx$ diverges.

15. If a function $y = f(x)$ is continuous and not bounded for all real x then the integral $\int_{a}^{\infty} f(x)dx$ diverges.

16. If a function $y = f(x)$ is continuous on $[1,\infty)$ and $\int_{1}^{\infty} f(x)dx$ converges then $\int_{1}^{\infty} |f(x)|dx$ also converges.

17. If the integral $\int_{a}^{\infty} f(x)dx$ converges and a function $y = g(x)$ is bounded then the integral $\int_{a}^{\infty} f(x)g(x)dx$ converges.

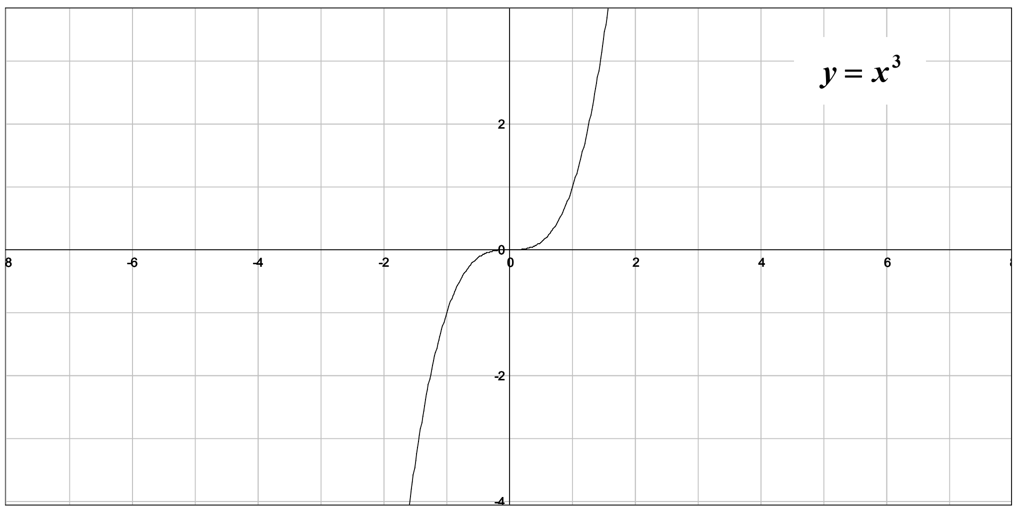
Suggested Solutions

1. Functions

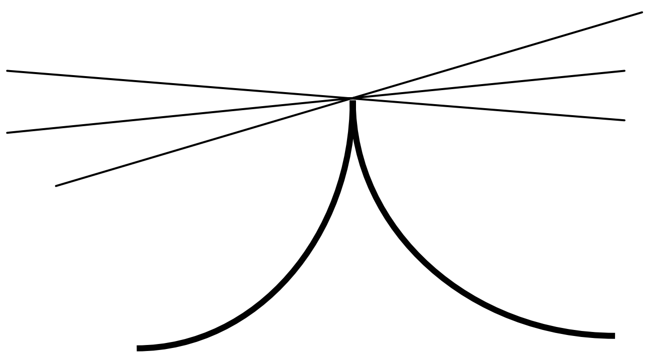
1. The tangent to a curve at a point is the line which touches the curve at that point but does not cross it there.

Counter-example.

a) The x-axis is the tangent line to the curve $y = x^3$ but it crosses the curve at the origin.



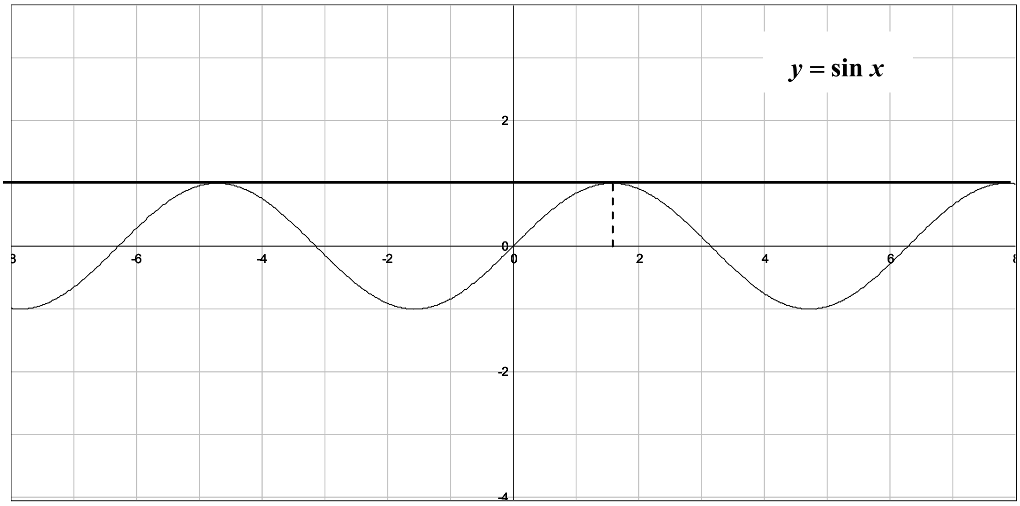
b) The three straight lines just touch and don't cross the curve below at the point but none of them is the tangent line to the curve at that point.



2. The tangent line to a curve at a point cannot touch the curve at infinitely many other points.

Counter-example.

The tangent line to the graph of the function $y = \sin x$ touches the curve at $x = \frac{\pi}{2}$ and infinitely many other points.



3. A quadratic function of x is one in which the highest power of x is two.

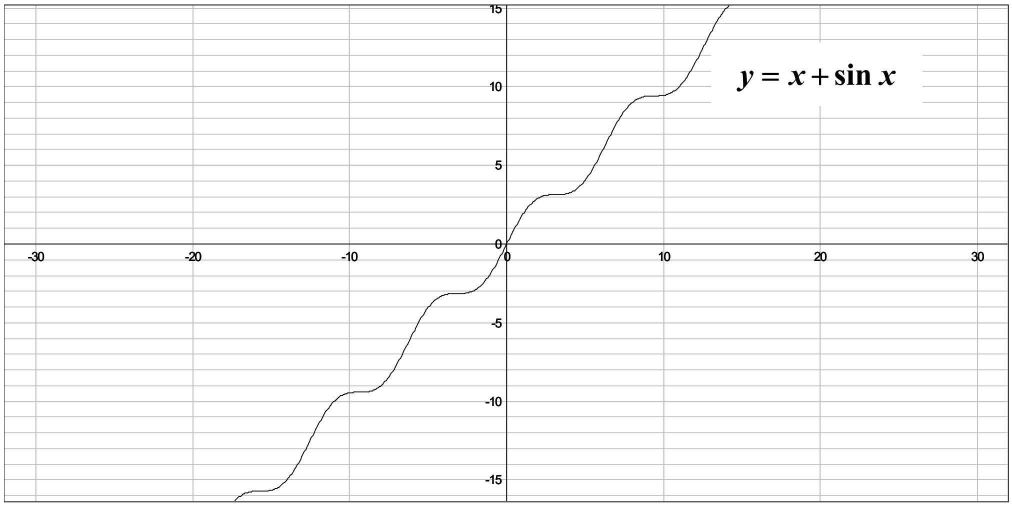
Counter-example.

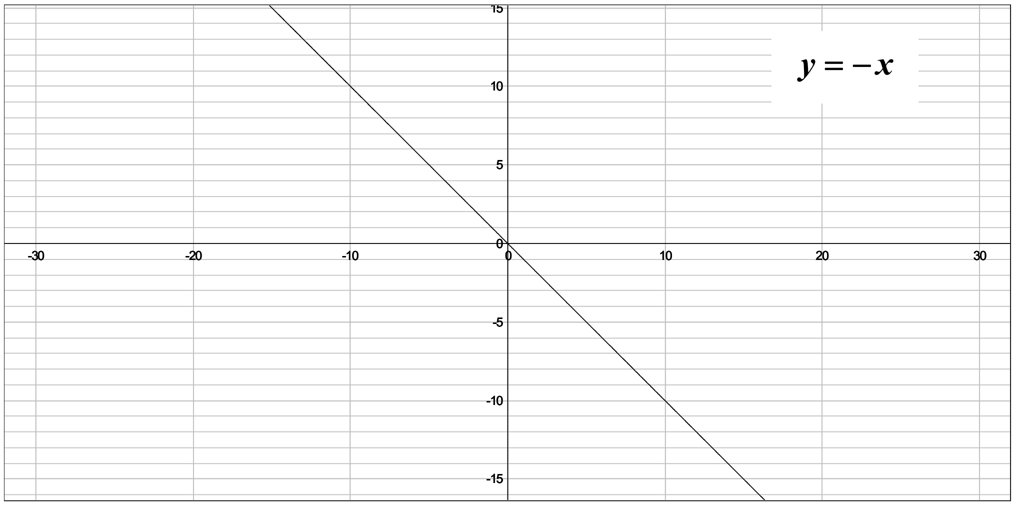
In both functions $y = x^2 + \sqrt{x}$ and $y = x^2 + x - \frac{1}{x}$ the highest power of x is two but neither is quadratic.

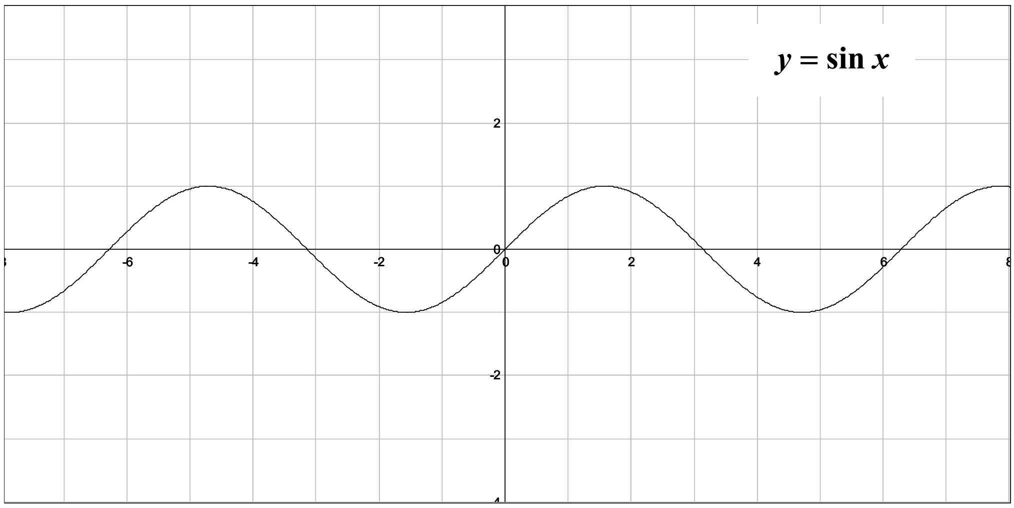
4. If both functions $y = f(x)$ and $y = g(x)$ are continuous and monotone on R then their sum $f(x) + g(x)$ is also monotone on R.

Counter-example.

$f(x) = x + \sin x$
$g(x) = - x$
Both functions $f(x)$ and $g(x)$ are monotone on R but their sum $f(x) + g(x) = \sin x$ is not monotone on R.



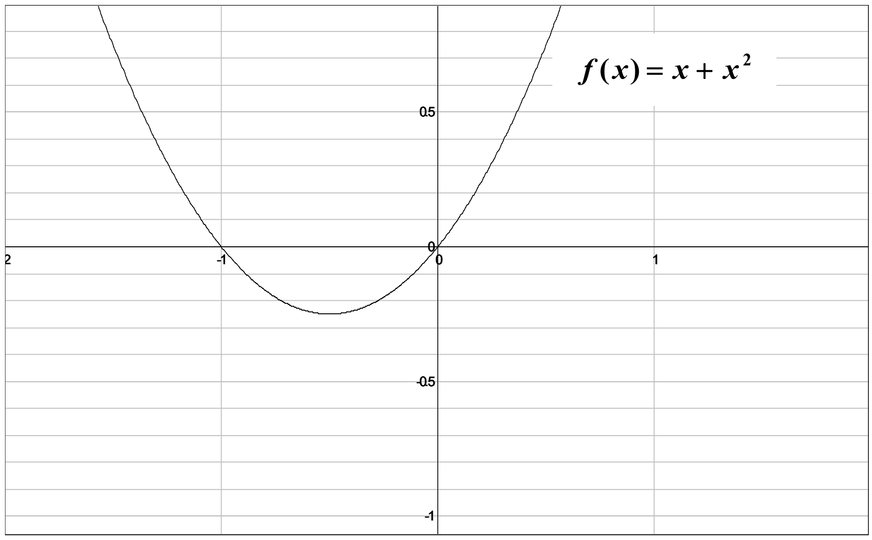


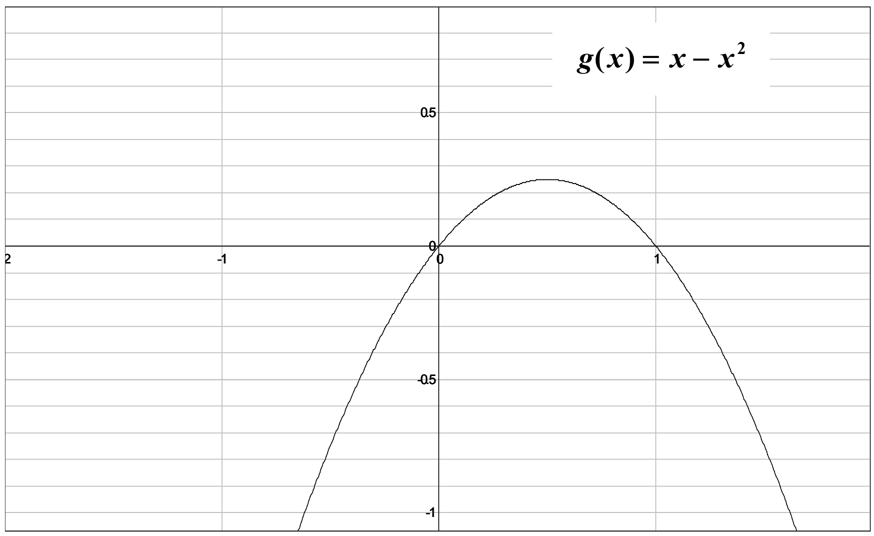


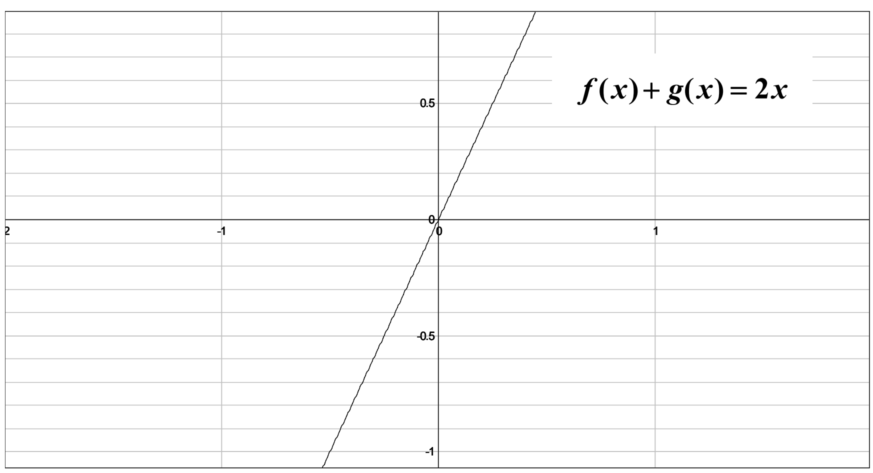
5. If both functions $y = f(x)$ and $y = g(x)$ are not monotone on R then their sum $f(x) + g(x)$ is not monotone on R.

Counter-example.

Both functions $f(x) = x + x^2$ and $g(x) = x - x^2$ are not monotone on R but their sum $f(x) + g(x) = 2x$ is monotone on R.



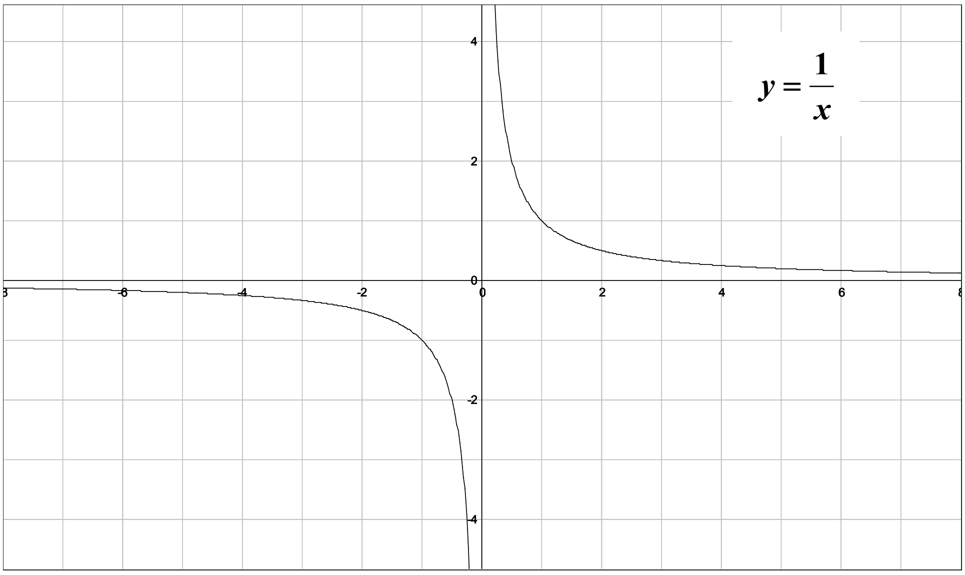




6. If a function $y = f(x)$ is continuous and decreasing for all positive x and $f(1)$ is positive then the function has exactly one root.

<u>Counter-example</u>.

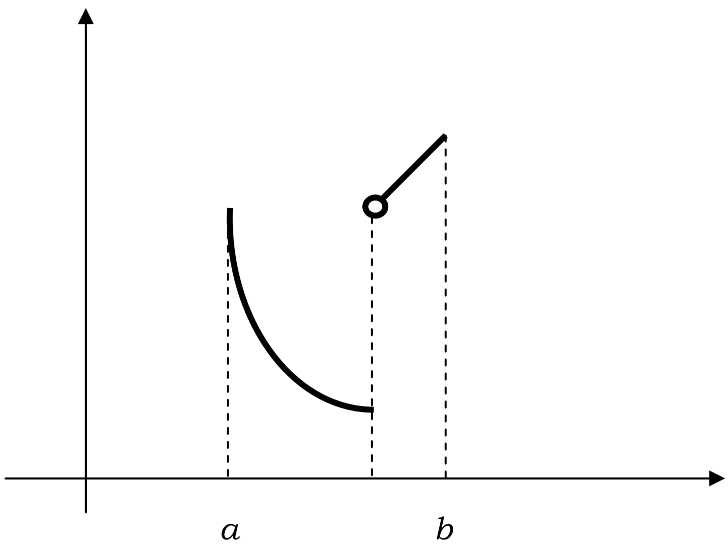
The function $y = \frac{1}{x}$ is continuous and decreasing for all positive x and $y(1) = 1 > 0$ but has no roots.



7. If a function $y = f(x)$ has an inverse function $x = f^{-1}(y)$ on (a,b) then the function $f(x)$ is either increasing or decreasing on (a,b).

<u>Counter-example</u>.

The function below is a one-to-one function and has an inverse function on (a,b) but it is neither increasing nor decreasing on (a,b).



8. A function $y = f(x)$ is bounded on R if for any $x \in R$ there is $M > 0$ such that $|f(x)| \le M$.

Counter-example.

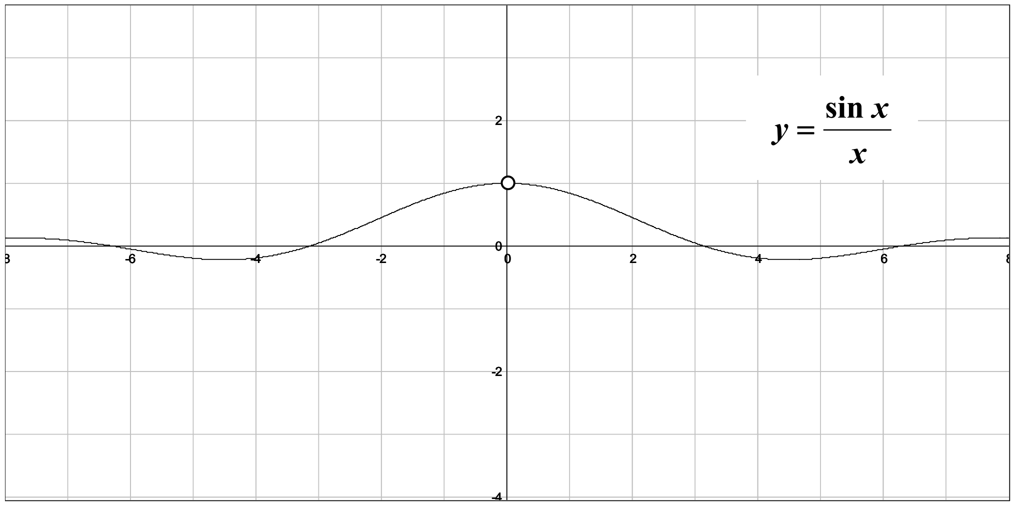
For the function $y = x^2$ for any value of x from R there is a number $M > 0$ ($M = x^2 + \varepsilon$, where $\varepsilon \ge 0$) such that $|f(x)| \le M$.

Comments. The order of words in this statement is very important. The correct definition of a function bounded on R differs only by the order of words: A function $y = f(x)$ is bounded on R if there is $M > 0$ such that for any $x \in R$ $|f(x)| \le M$.

9. If $g(a) = 0$ then the function $F(x) = \dfrac{f(x)}{g(x)}$ has a vertical asymptote at the point $x = a$.

Counter-example.

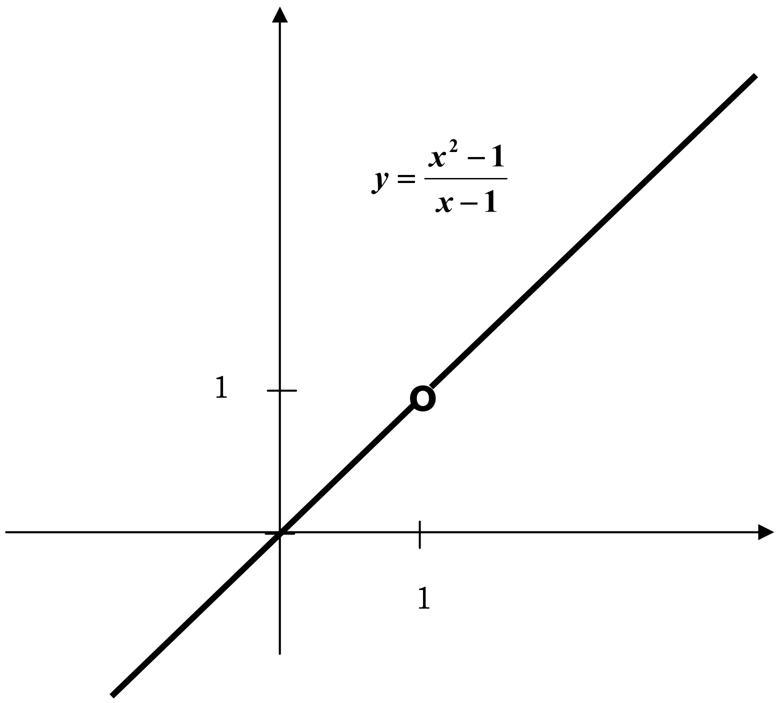
The function $y = \dfrac{\sin x}{x}$ doesn't have a vertical asymptote at the point $x = 0$.



10. If $g(a) = 0$ then the *rational* function $R(x) = \frac{f(x)}{g(x)}$ (both $f(x)$ and $g(x)$ are polynomials) has a vertical asymptote at the point $x = a$.

<u>Counter-example</u>.

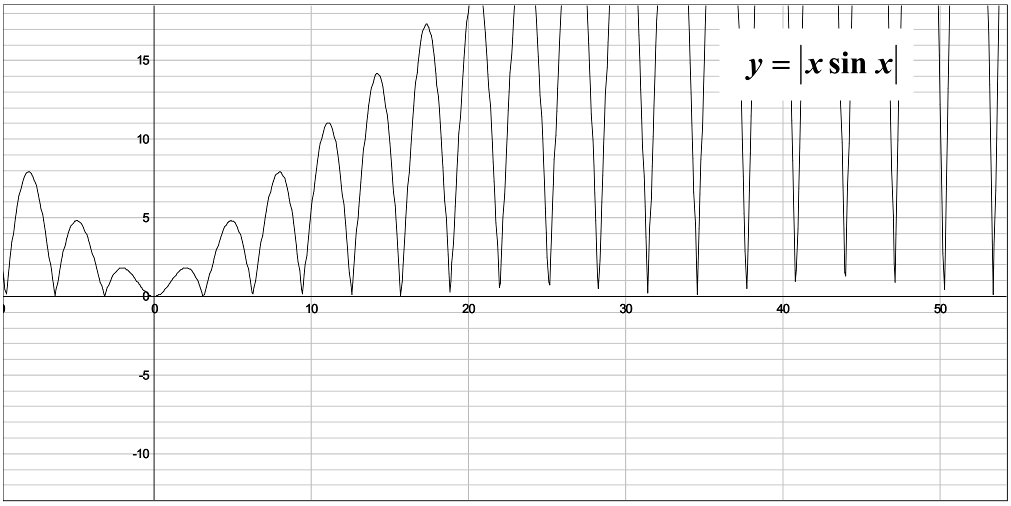
The rational function $y = \frac{x^2 - 1}{x - 1}$ doesn't have a vertical asymptote at the point $x = 1$.



11. If a function $y = f(x)$ is unbounded and non-negative for all real x then it cannot have roots x_n such that $x_n \to \infty$ as $n \to \infty$.

Counter-example.

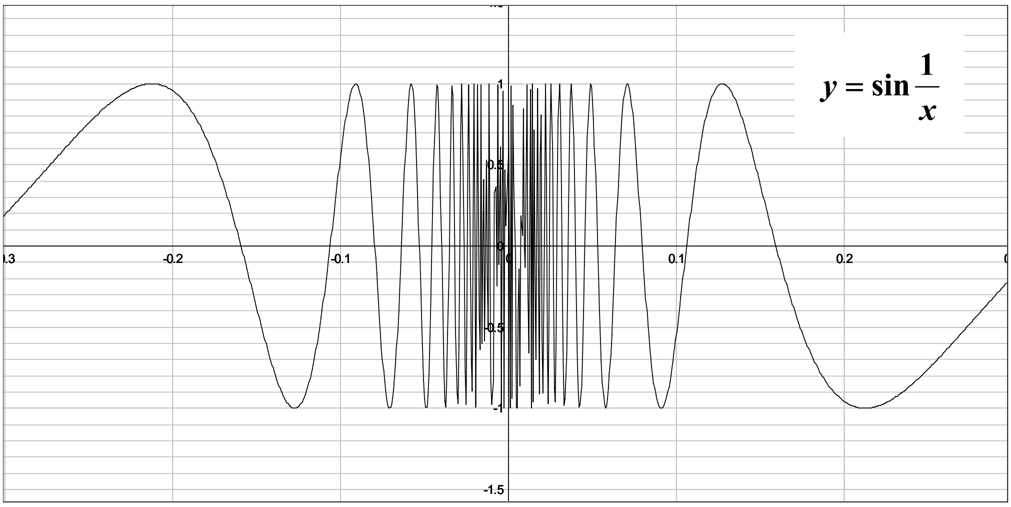
The function $y = |x \sin x|$ has infinitely many roots x_n such that $x_n \to \infty$ as $n \to \infty$.



12. A function $y = f(x)$ defined on $[a,b]$ such that its graph doesn't contain any pieces of a horizontal straight line cannot take its extreme value infinitely many times on $[a,b]$.

Counter-example.

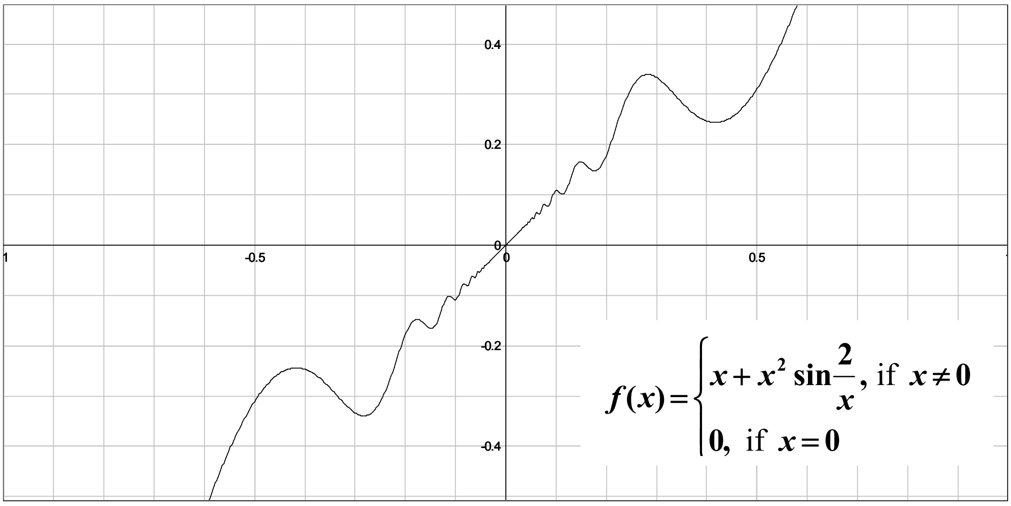
The function $y = \sin \frac{1}{x}$ takes its absolute maximum value (=1) and its absolute minimum value (=-1) infinitely many times on any closed interval containing zero.



13. If a function $y = f(x)$ is continuous and increasing at the point $x = a$ then there is a neighbourhood $(x-\delta, x+\delta), \delta > 0$ where the function is also increasing.

Counter-example.

The function $f(x) = \begin{cases} x + x^2 \sin\frac{2}{x}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases}$ is increasing at the point $x = 0$ but it is not increasing in any neighbourhood $(-\delta, \delta)$, where $\delta > 0$.



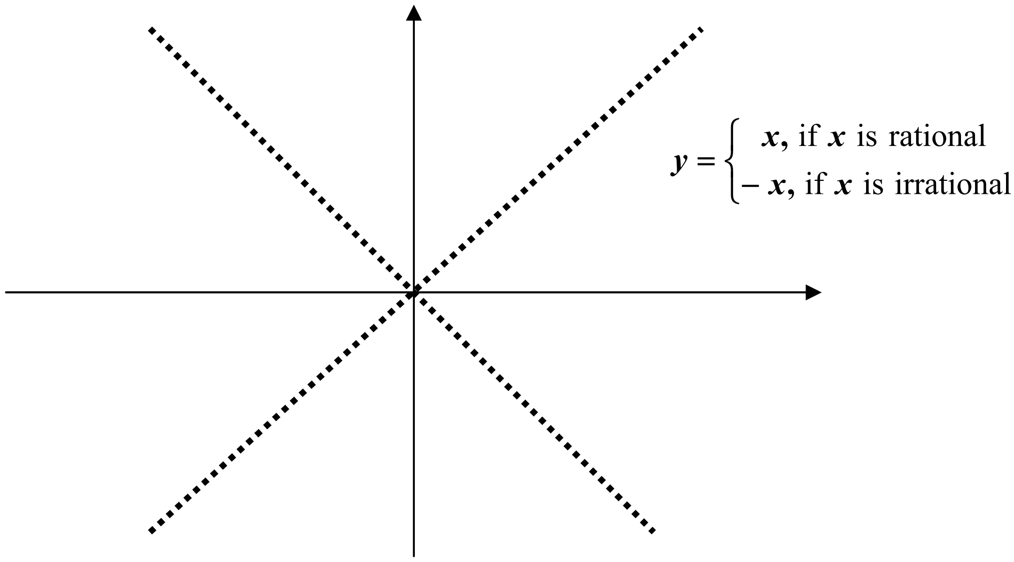
Comments. The definition of a function increasing at a point is:
A function $y = f(x)$ is said to be *increasing at the point* $x = a$ if in a certain neighbourhood $(a-\delta, a+\delta), \delta > 0$ the following is true:
if $x < a$ then $f(x) < f(a)$ and if $x > a$ then $f(x) > f(a)$.

14. If a function is not monotone then it doesn't have an inverse function.

Counter-example.

The function $y = \begin{cases} x, \text{ if } x \text{ is rational} \\ -x, \text{ if } x \text{ is irrational} \end{cases}$ is not monotone but it has the inverse function $x = \begin{cases} y, & \text{if } y \text{ is rational} \\ -y, & \text{if } y \text{ is irrational.} \end{cases}$

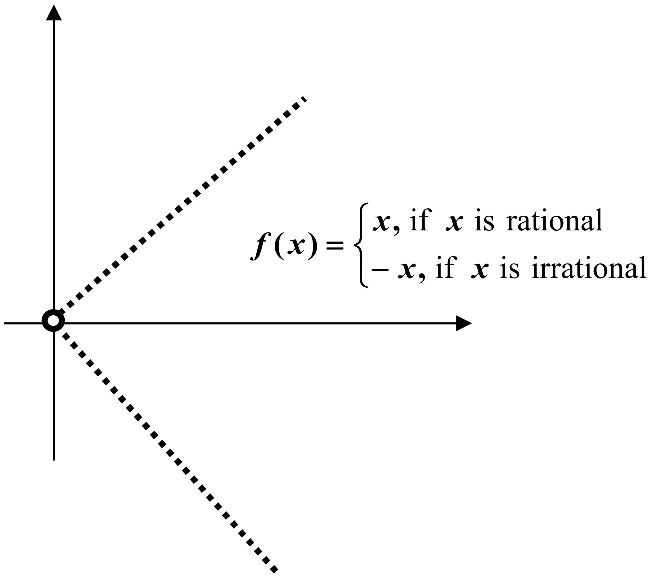
It is impossible to draw the graph of such a function but a rough sketch gives an idea of its behaviour:

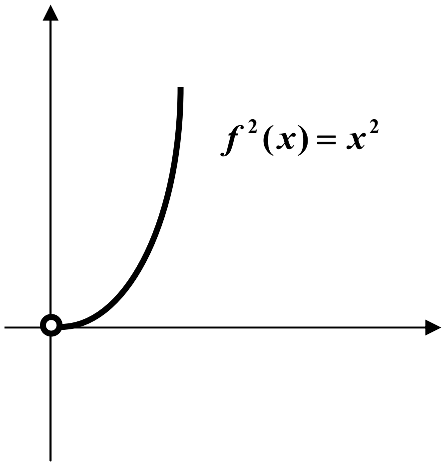


15. If a function is not monotone on (a,b) then its square cannot be monotone on (a,b).

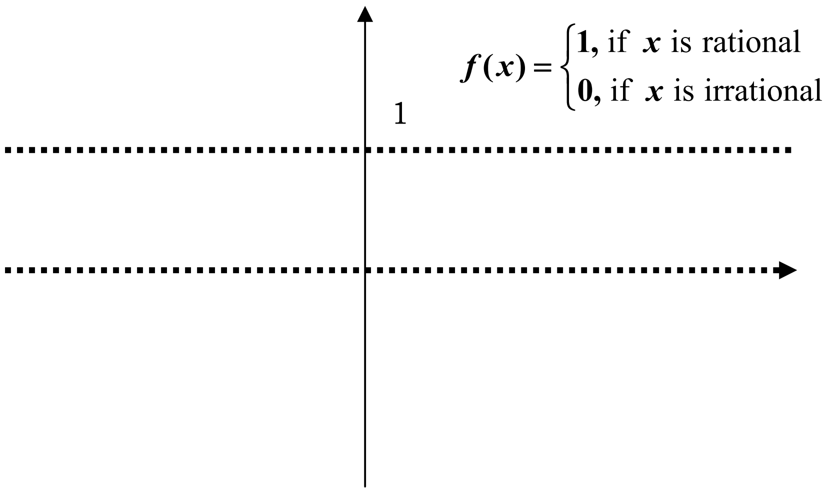
<u>Counter-example</u>.

The function $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$ defined on $(0,\infty)$ is not monotone but its square $f^2(x) = x^2$ is monotone on $(0,\infty)$.
It is impossible to draw the graph of the function $y = f(x)$ but the sketch below gives an idea of its behaviour.





Comments. The functions in counter-examples 14 and 15 may seem artificial and without practical use at first. Nevertheless, the Dirichlet function $f(x) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational} \end{cases}$, which is very similar to the functions in counter-examples 14 and 15, can be represented analytically as a limit of cosine functions that have many practical applications: $f(x) = \lim_{k\to\infty} \lim_{n\to\infty} (\cos(k!\pi x))^{2n}$.

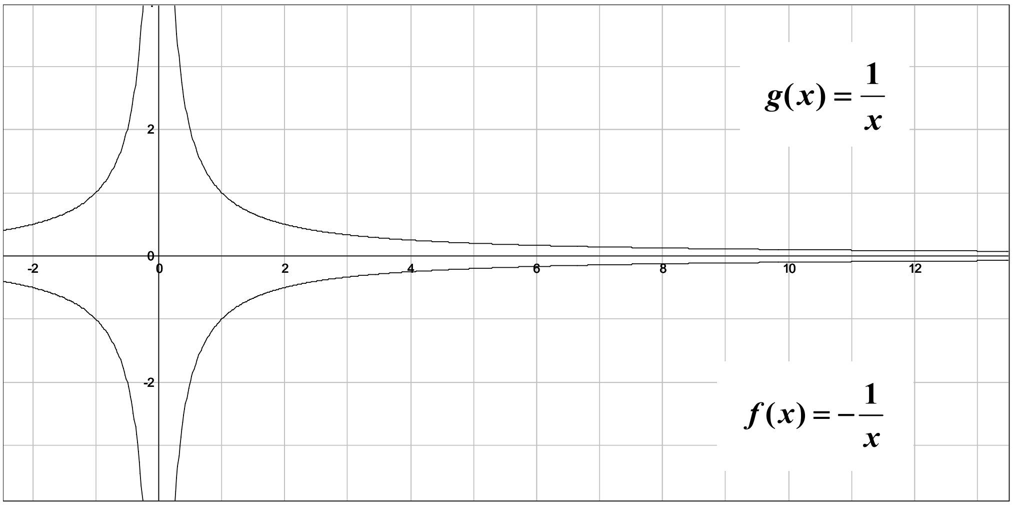


2. Limits

1. If $f(x) < g(x)$ for all $x > 0$ and both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ exist then $\lim_{x\to\infty} f(x) < \lim_{x\to\infty} g(x)$.

<u>Counter-example</u>.

For the functions $f(x) = -\frac{1}{x}$ and $g(x) = \frac{1}{x}$ $f(x) < g(x)$ for all $x > 0$ but $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$.



2. The following definitions of a non-vertical asymptote are equivalent:

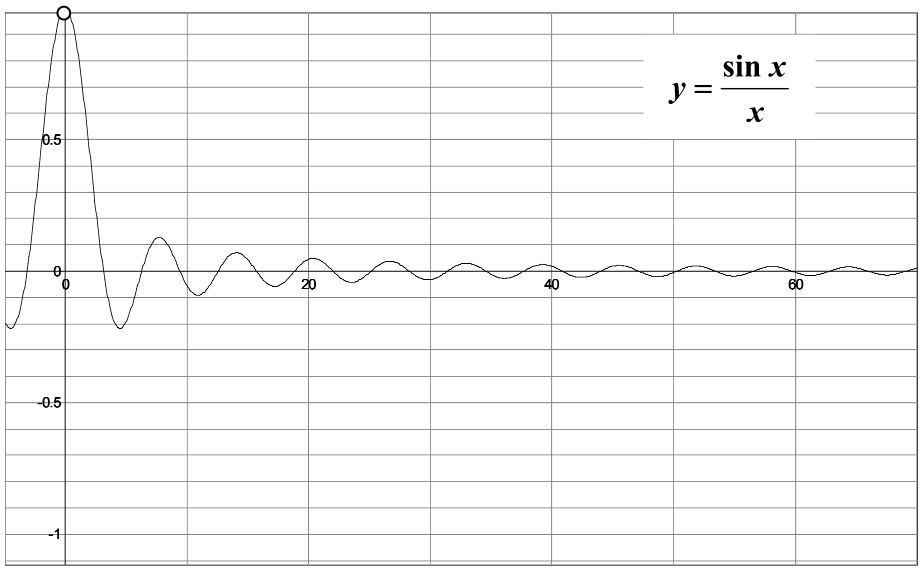
a) The straight line $y = mx + c$ is called a non-vertical asymptote to a curve $f(x)$ as x tends to infinity if $\lim_{x\to\infty}(f(x) - (mx + c)) = 0$.

b) A straight line is called a non-vertical asymptote to a curve as x tends to infinity if the curve gets closer and closer to the straight line (as close as we like) as x tends to infinity but doesn't touch or cross it.

Counter-example.

As x tends to infinity the function $y = \frac{\sin x}{x}$ gets closer to the x-axis from above and below and $\lim_{x \to \infty}(\frac{\sin x}{x} - 0) = 0$. According to the first definition the x-axis is the non-vertical asymptote of the function $y = \frac{\sin x}{x}$, but its graph crosses the x-axis infinitely many times, so the definitions a) and b) are not equivalent.

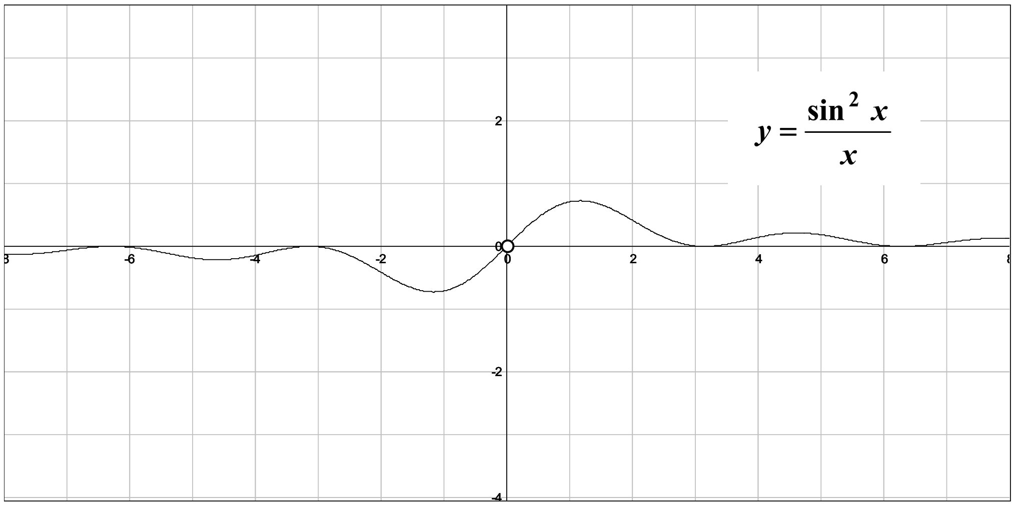
Comments. The correct definition is a). The idea of an asymptotic behaviour is getting closer to a (non-vertical) straight line but this doesn't exclude touching or crossing it.



3. The tangent line to a curve at a certain point that touches the curve at infinitely many other points cannot be a non-vertical asymptote to this curve.

Counter-example.

The tangent line $y = 0$ to the curve $y = \frac{\sin^2 x}{x}$ at $x = \pi$ touches the curve at infinitely many other points and is a non-vertical asymptote to this curve.



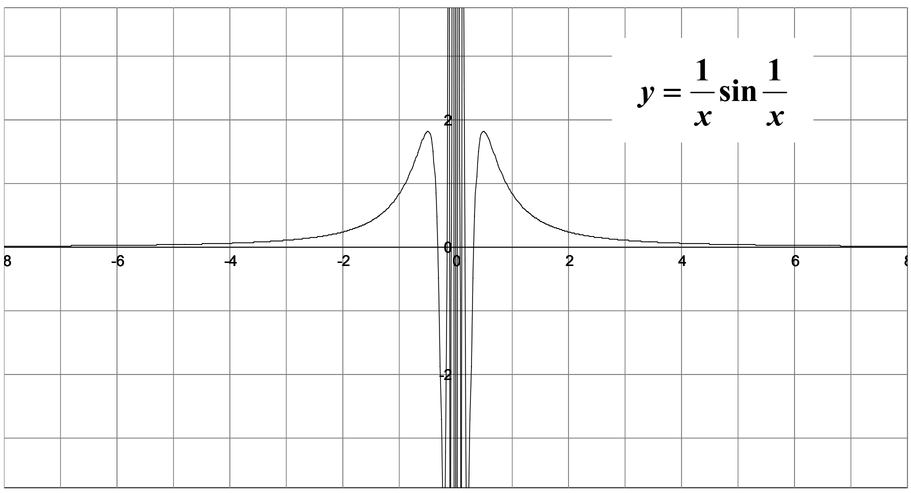
4. The following definitions of a vertical asymptote are equivalent:

a) The straight line $x = a$ is called a vertical asymptote for a function $y = f(x)$ if $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

b) The straight line $x = a$ is called a vertical asymptote for the function $y = f(x)$ if there are infinitely many values of $f(x)$ that can be made arbitrarily large or arbitrarily small as x gets closer to a from either side of a.

<u>Counter-example</u>.

There are infinitely many values of the function $y = \frac{1}{x} \times \sin\frac{1}{x}$ that can be made arbitrarily large or small as x gets closer to 0 but the straight line $x = 0$ is not a vertical asymptote of this function.

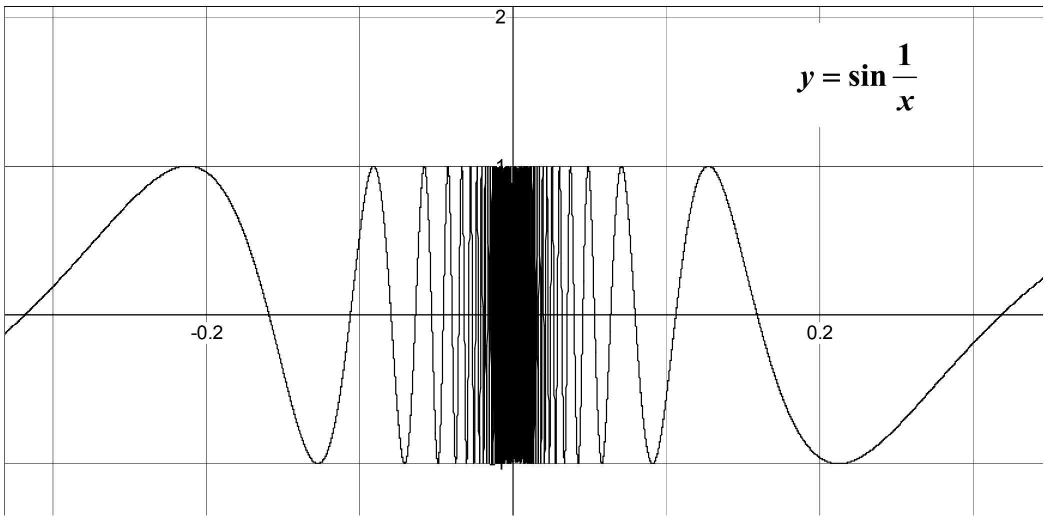


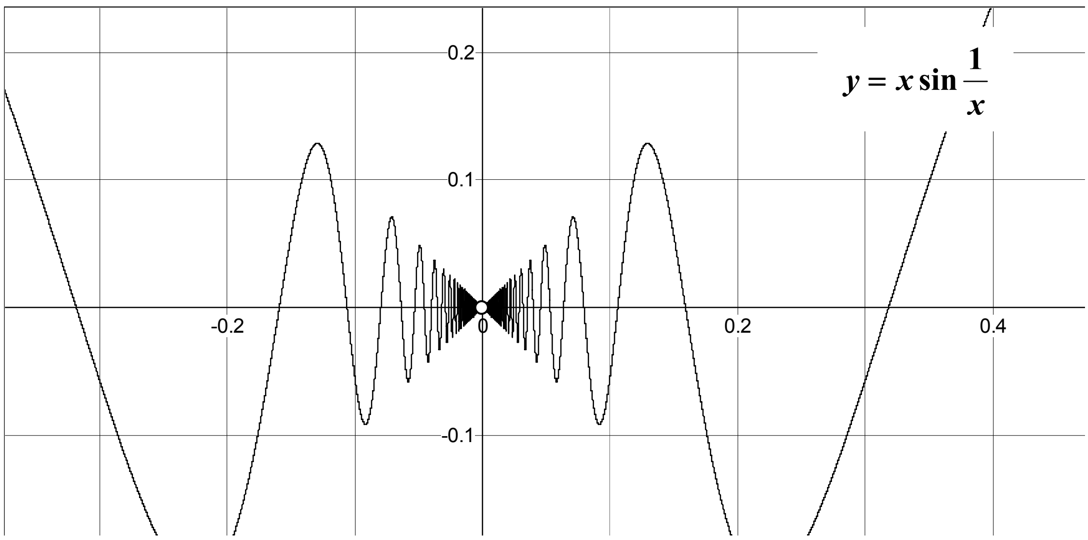
<u>Comments</u>. The correct definition is a).

5. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does't exist because of oscillation of $g(x)$ near $x = a$ then $\lim_{x\to a}(f(x)\times g(x))$ doesn't exist.

<u>Counter-example</u>.

For the function $f(x) = x$ the limit $\lim_{x\to 0} x = 0$ and for the function $g(x) = \sin\frac{1}{x}$ the limit $\lim_{x\to 0}\sin\frac{1}{x}$ doesn't exist because of oscillation of $g(x)$ near $x = 0$, but $\lim_{x\to 0}(f(x)\times g(x)) = \lim_{x\to 0}(x\sin\frac{1}{x}) = 0$.

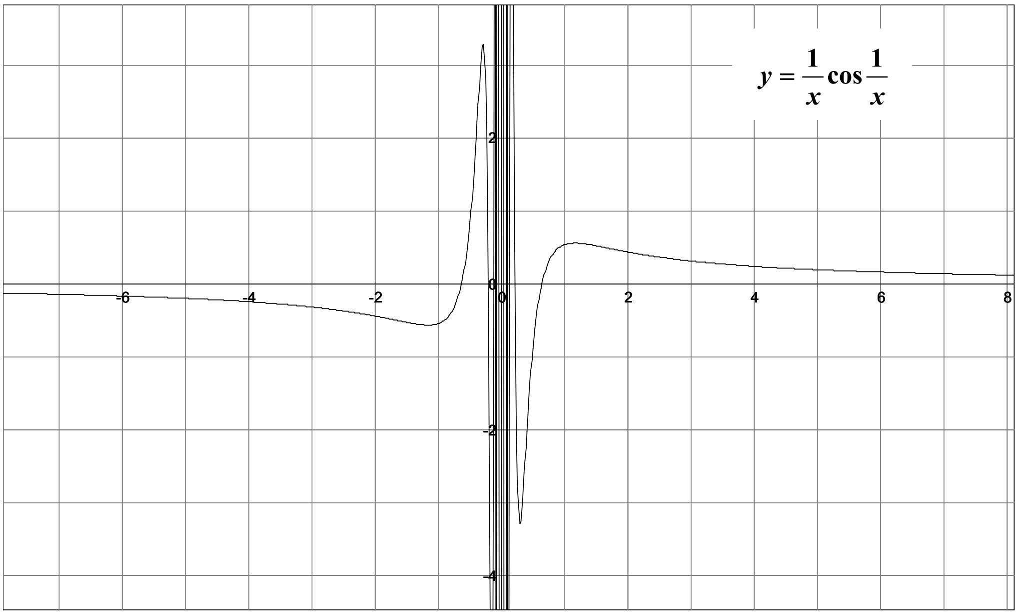




6. If a function $y = f(x)$ is not bounded in any neighbourhood of the point $x = a$ then either $\lim\limits_{x\to a+}|f(x)| = \infty$ or $\lim\limits_{x\to a^-}|f(x)| = \infty$.

Counter-example.

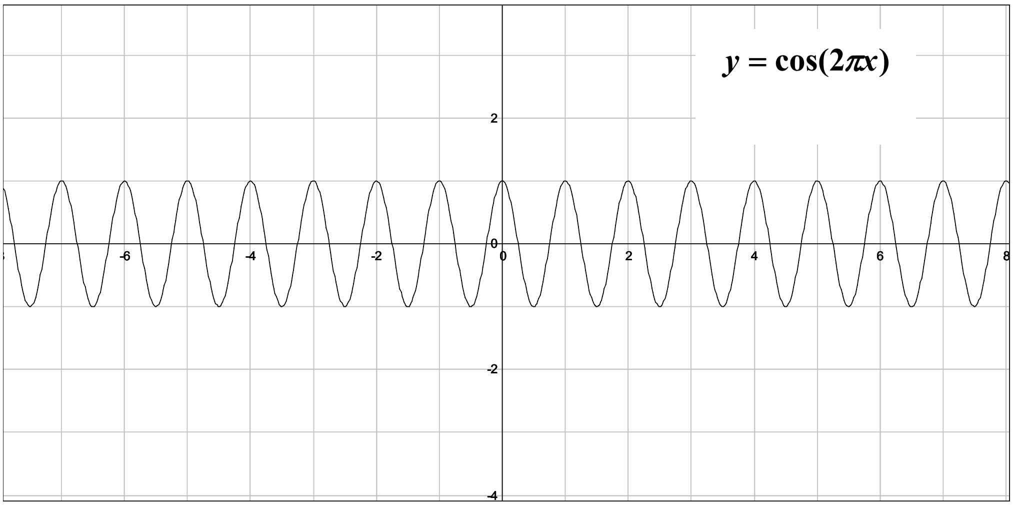
The function $f(x) = \frac{1}{x}\cos\frac{1}{x}$ is not bounded in any neighbourhood of the point $x = 0$ but neither $\lim\limits_{x\to 0^+}\left|\frac{1}{x}\cos\frac{1}{x}\right|$ nor $\lim\limits_{x\to 0^-}\left|\frac{1}{x}\cos\frac{1}{x}\right|$ exist.



7. If a function $y = f(x)$ is continuous for all real x and $\lim\limits_{n\to\infty} f(n) = A$ then $\lim\limits_{x\to\infty} f(x) = A$.

Counter-example.

For the continuous function $y = \cos(2\pi x)$ the limit $\lim\limits_{n\to\infty}\cos(2\pi n)$ equals 1 because $\cos(2\pi n) = 1$ for any natural n but $\lim\limits_{x\to\infty}\cos(2\pi x)$ does not exist.



<u>Comments</u>. Statement 6 is the converse of the true statement: $\lim_{x\to\infty} f(x) = A \Rightarrow \lim_{n\to\infty} f(n) = A$.

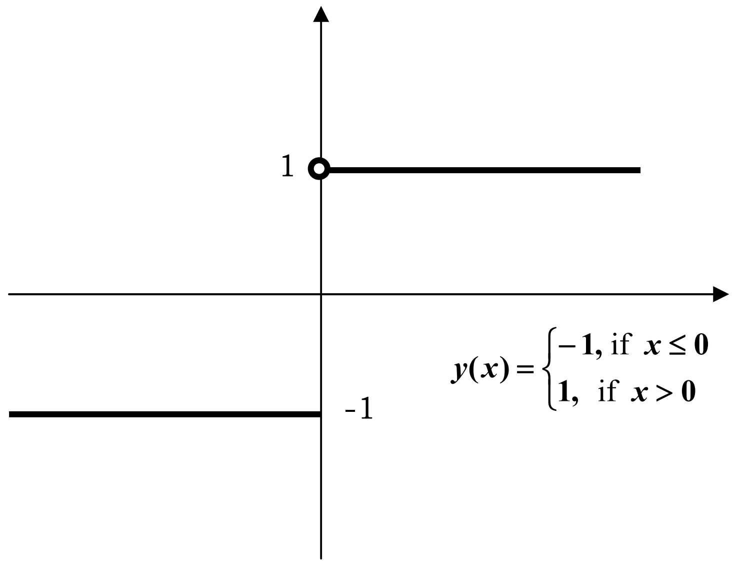
3. Continuity

1. If the absolute value of the function $y = f(x)$ is continuous on (a,b) then the function is also continuous on (a,b).

Counter-example.

The absolute value of the function

$y(x) = \begin{cases} -1, \text{ if } x \le 0 \\ 1, \text{ if } x > 0 \end{cases}$ is $|y(x)| = 1$ for all real x and it is continuous, but the function $y(x)$ is discontinuous.



2. If both functions $y = f(x)$ and $y = g(x)$ are discontinuous at $x = a$ then $f(x) + g(x)$ is also discontinuous at $x = a$.

Counter-example.

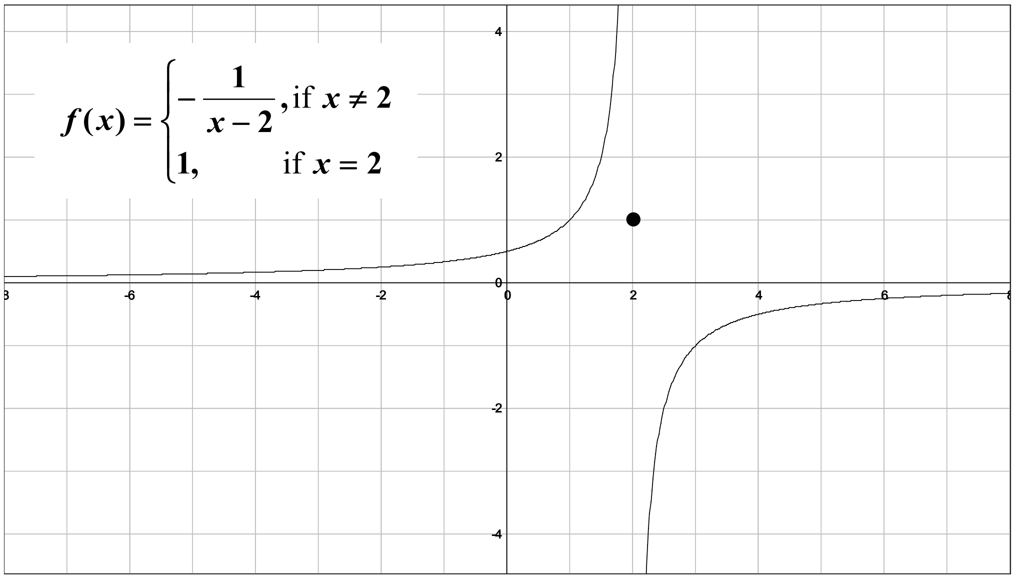
$$f(x) = -\frac{1}{x-a}, \text{ if } x \ne a$$

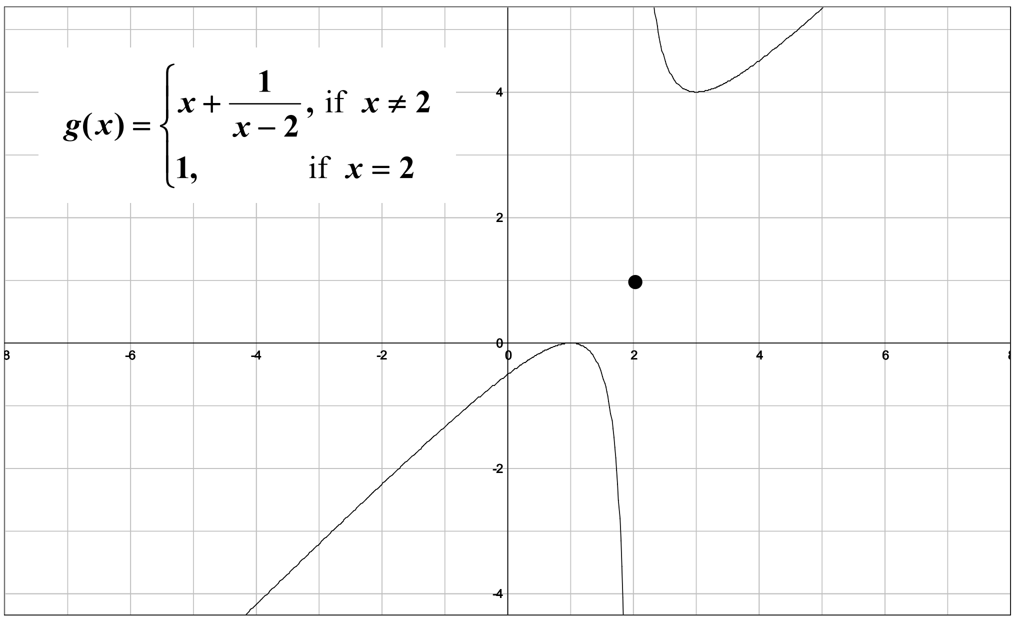
$$g(x) = x + \frac{1}{x-a}, \text{ if } x \ne a$$

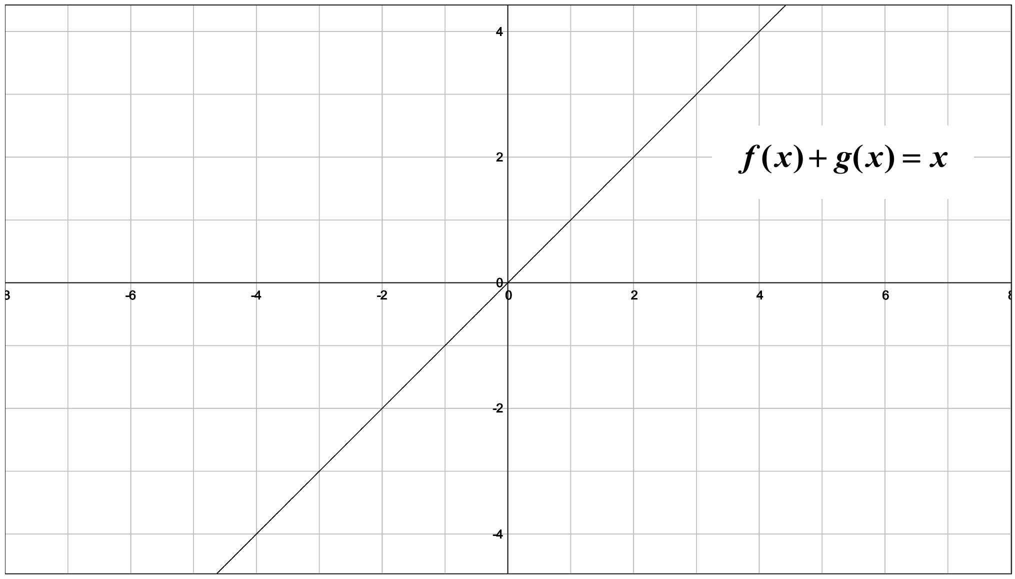
$$f(x) = g(x) = \frac{a}{2}, \text{ if } x = a$$

Both functions $f(x)$ and $g(x)$ are discontinuous at $x = a$ but the function $f(x)+g(x)=\begin{cases} x, & \text{if } x \neq a \\ a, & \text{if } x = a \end{cases}$ is continuous at $x = a$.

For example, if $a = 2$:



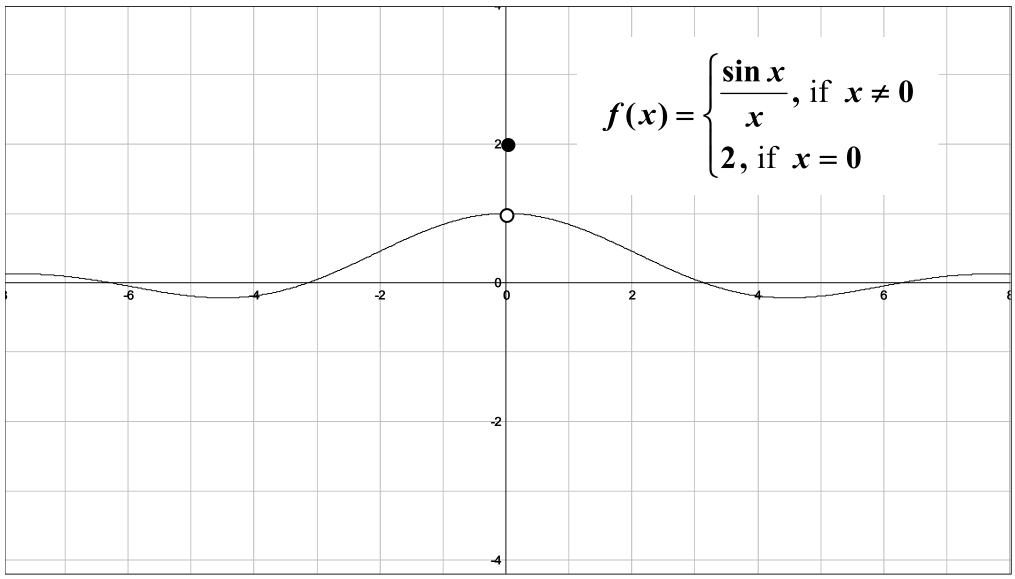


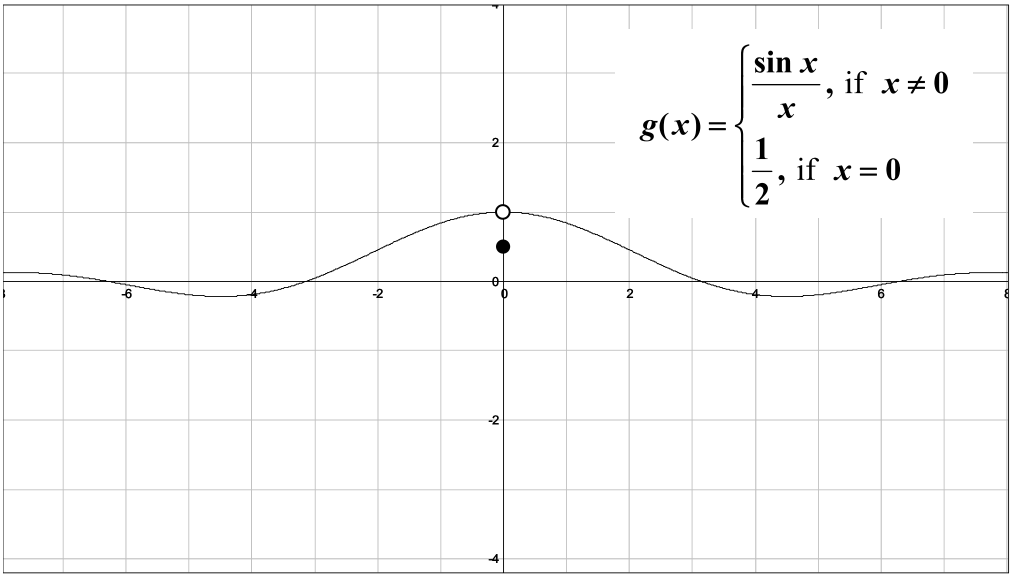


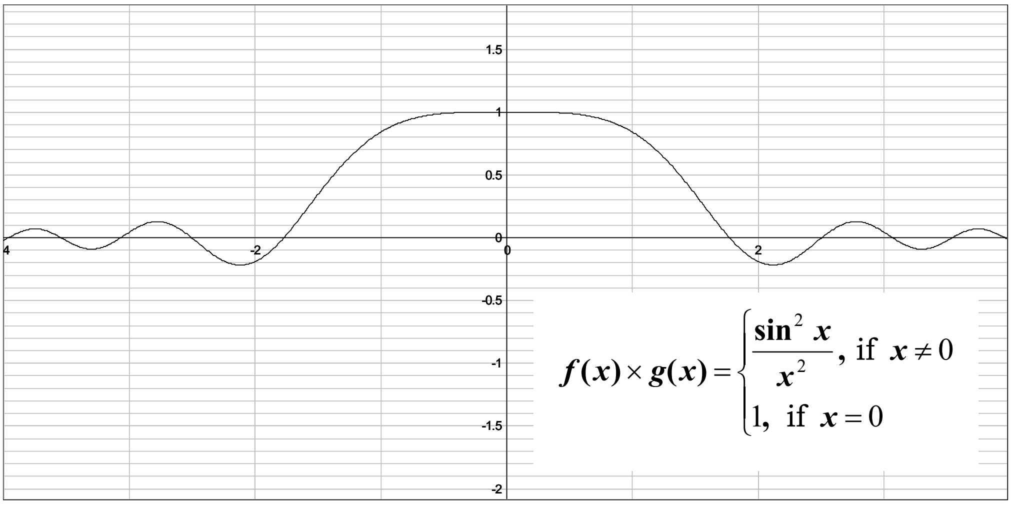
3. If both functions $y = f(x)$ and $y = g(x)$ are discontinuous at $x = a$ then $f(x)\times g(x)$ is also discontinuous at $x = a$.

Counter-example.

Both functions $f(x)=\begin{cases}\dfrac{\sin x}{x}, \text{ if } x\neq 0\\ 2, \text{ if } x=0\end{cases}$ and $g(x)=\begin{cases}\dfrac{\sin x}{x}, \text{ if } x\neq 0\\ \dfrac{1}{2}, \text{ if } x=0\end{cases}$ are discontinuous at the point $x = 0$ but their product $f(x)\times g(x)=\begin{cases}\dfrac{\sin^2 x}{x^2}, \text{ if } x\neq 0\\ 1, \text{ if } x=0\end{cases}$ is continuous at the point $x = 0$.



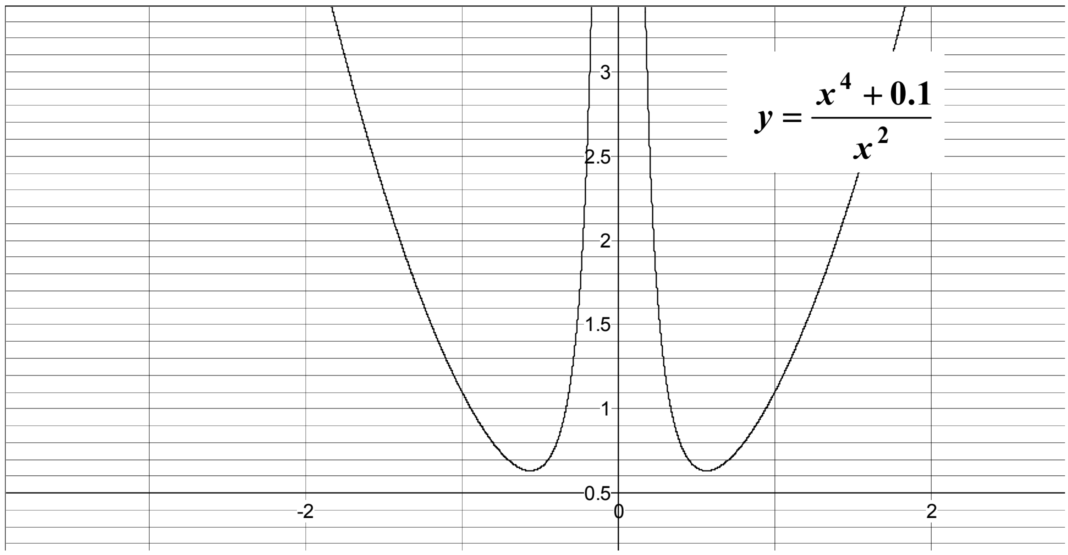


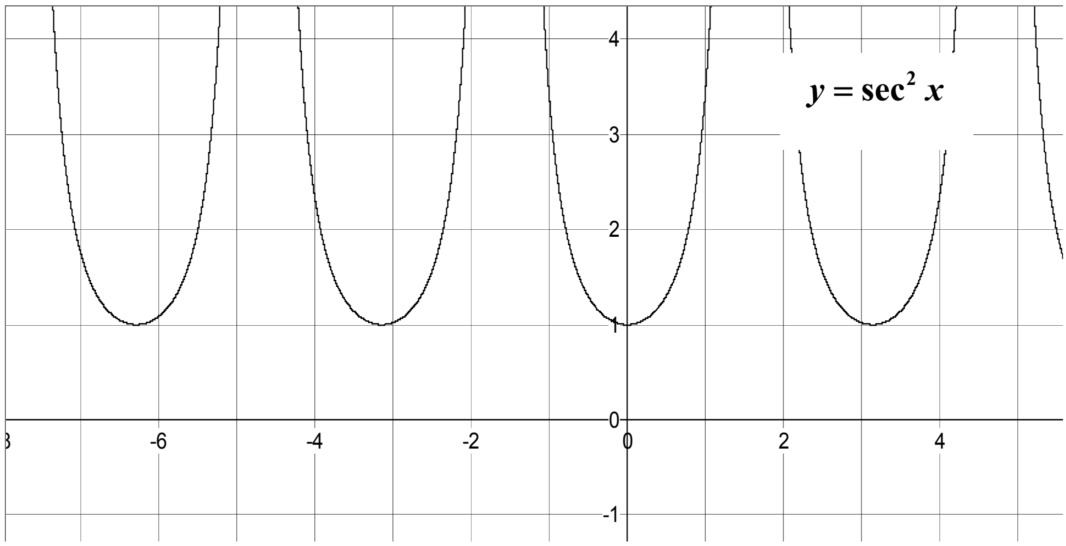


4. A function always has a local maximum between any two local minima.

<u>Counter-example</u>.

The functions $y=\dfrac{x^4+0.1}{x^2}$ and $y=\sec^2 x$ have no maximum between two local minima:

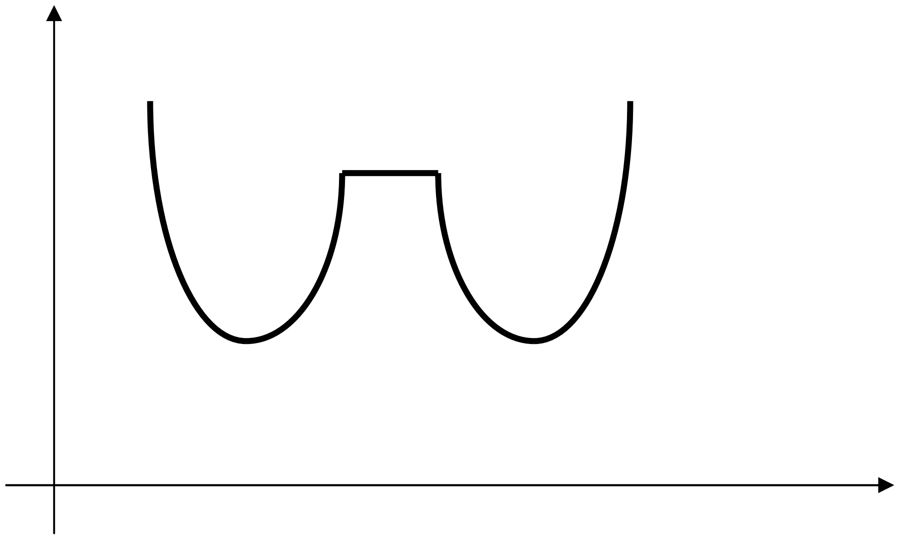




5. For a continuous function there is always a local maximum between any two local minima.

Counter-example.

The continuous function below doesn't have a local maximum between its two local minima.

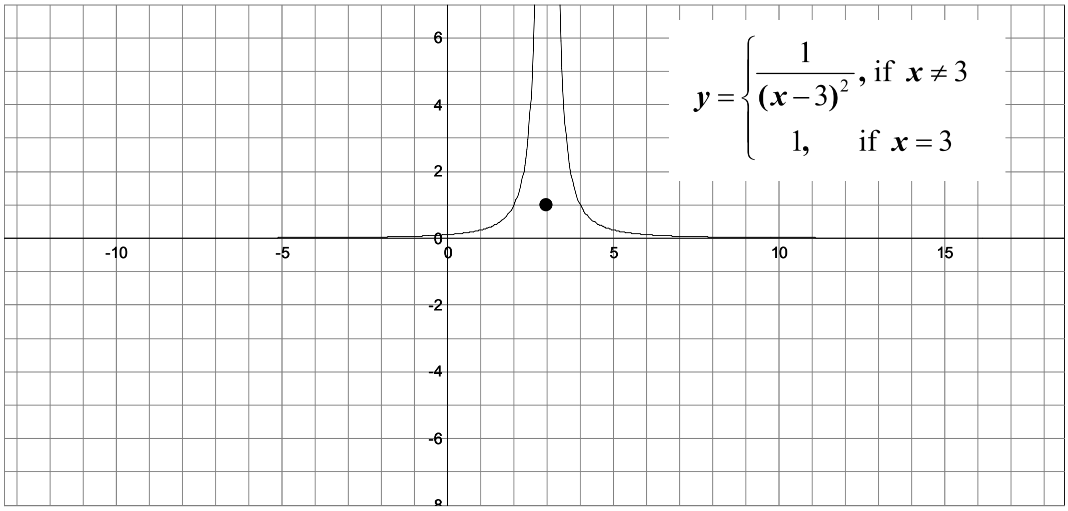


Comments. A *strict* inequality in the definition of a local maximum is accepted here: a function $y = f(x)$ has a local maximum at the point $x = a$ if $f(a) > f(x)$ for all x within a certain neighbourhood $(a-\delta, a+\delta), \delta > 0$ of the point $x = a$. Otherwise in the above graph we have to treat each point of the line segment as a local maximum.

6. If a function is defined in a certain neighbourhood of point $x = a$ including the point itself and is increasing on the left from $x = a$ and decreasing on the right from $x = a$, then there is a local maximum at $x = a$.

Counter-example.

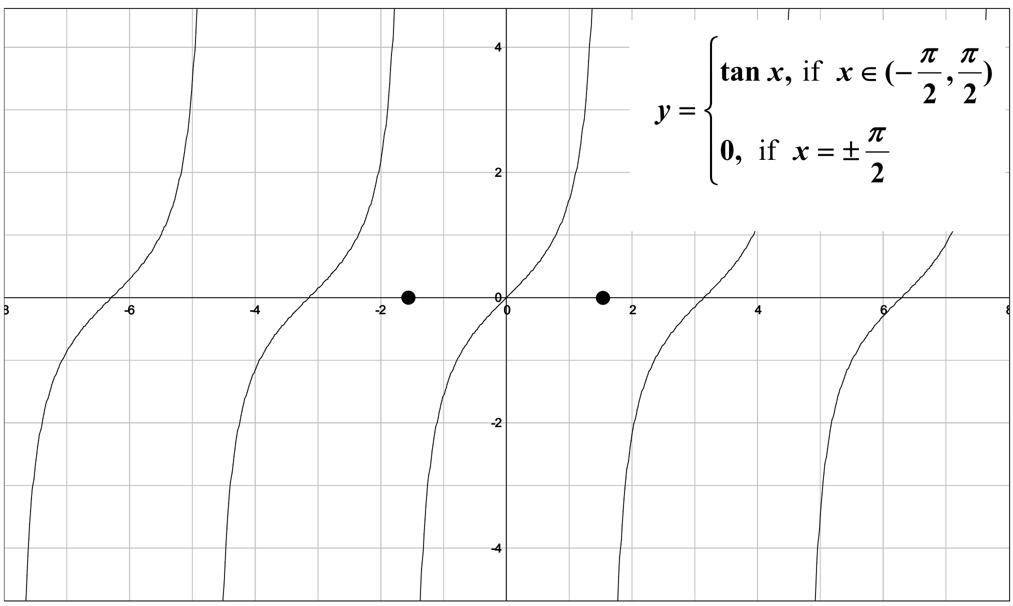
The function $y = \begin{cases} \dfrac{1}{(x-3)^2}, & \text{if } x \neq 3 \\ 1, & \text{if } x = 3 \end{cases}$ is defined for all real x, increasing on the left from the point $x = 3$ and decreasing on the right from the point $x = 3$ but has no a local maximum at the point $x = 3$.



7. If a function is defined on $[a,b]$ and continuous on (a,b) then it takes its extreme values on $[a,b]$.

<u>Counter-example</u>.

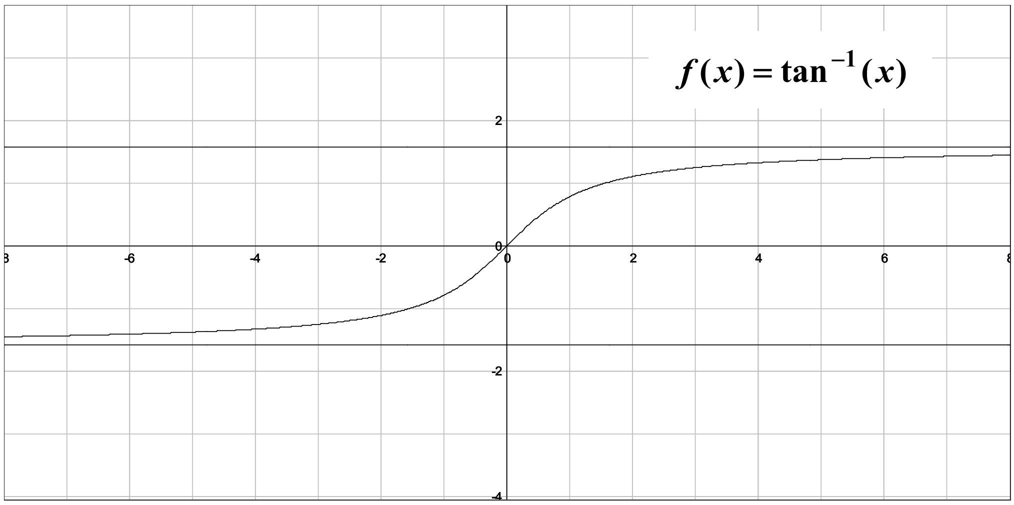
The function $y=\begin{cases}\tan x, \text{ if } x\in(-\frac{\pi}{2},\frac{\pi}{2})\\ 0, \text{ if } x=\pm\frac{\pi}{2}\end{cases}$ is defined on $[-\frac{\pi}{2},\frac{\pi}{2}]$ and continuous on $(-\frac{\pi}{2},\frac{\pi}{2})$ but it has no extreme values on $[-\frac{\pi}{2},\frac{\pi}{2}]$.



8. Every continuous and bounded function on $(-\infty,\infty)$ takes on its extreme values.

<u>Counter-example</u>.

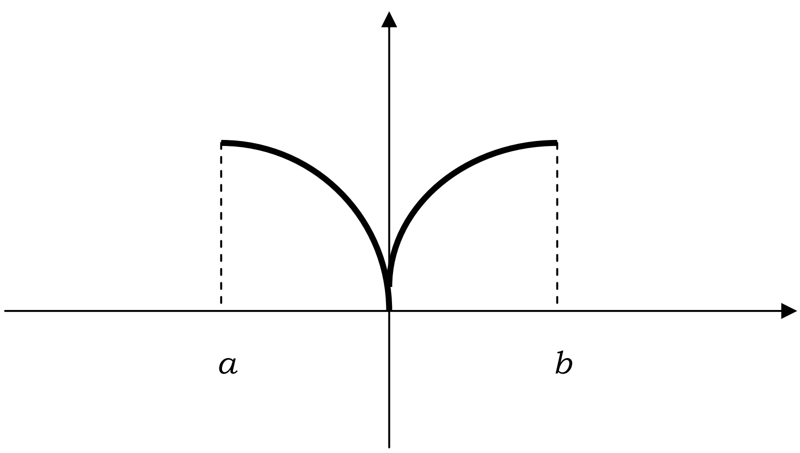
The function $f(x) = \tan^{-1}(x)$ is continuous and bounded on $(-\infty,\infty)$ but takes no extreme values.



9. If a function $y = f(x)$ is continuous on $[a,b]$, the tangent line exists at all points on its graph and $f(a) = f(b)$ then there is a point c in (a,b) such that the tangent line at the point $(c,f(c))$ is horizontal.

<u>Counter-example</u>.

The function $y = f(x)$ below is continuous on $[a,b]$, the tangent line exists at all points on the graph and $f(a) = f(b)$ but there is no point c in (a,b) such that the tangent line at the point $(c,f(c))$ is horizontal.



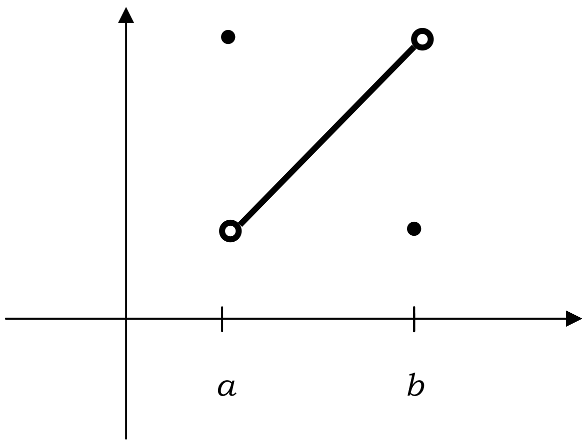
10. If on the closed interval $[a,b]$ a function is:

   a. bounded;
   b. takes its maximum and minimum values;
   c. takes all its values between the maximum and minimum values;

then this function is continuous on $[a,b]$.

<u>Counter-example</u>.

The function below satisfies the three conditions above, but is not continuous on $[a,b]$.



11. If on the closed interval $[a,b]$ a function is:

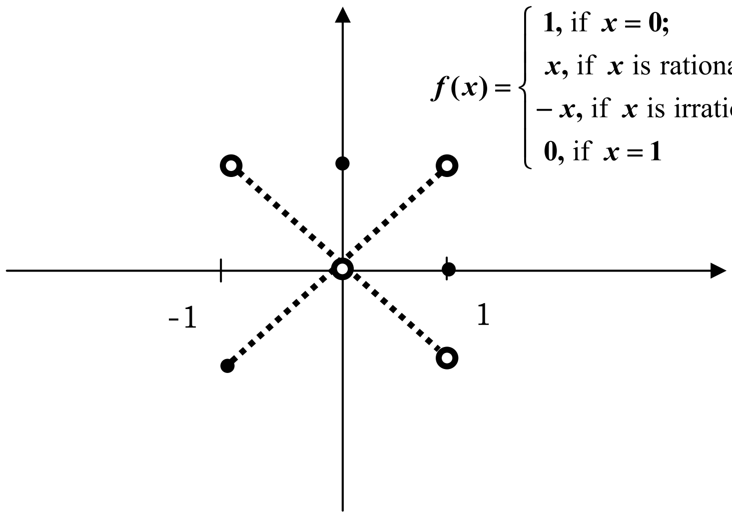
   a. bounded;
   b. takes its maximum and minimum values;
   c. takes all its values between the maximum and minimum values;

then this function is continuous at some points or subintervals on $[a,b]$.

<u>Counter-example</u>.

The function below satisfies all three conditions above but it is discontinuous at *every* point on [-1,1]. It is impossible to draw the graph of the function $y = f(x)$ but the sketch below gives an idea of its behaviour.

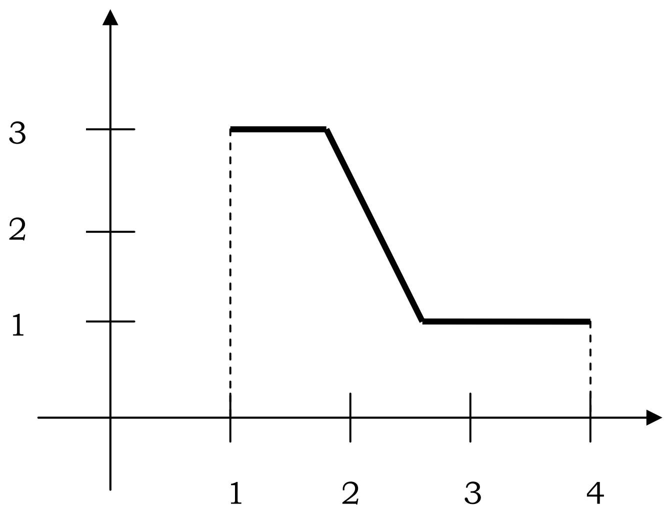
$$f(x)=\begin{cases}\mathbf{1}, \text{ if } x=0; \\ x, \text{ if } x \text{ is rational}, x\neq 0,\ x\neq 1; \\ -x, \text{ if } x \text{ is irrational}, x\neq 0,\ x\neq 1,\ x\neq -1 \\ 0, \text{ if } x=1\end{cases}$$



12. If a function is continuous on $[a,b]$ then it cannot take its absolute maximum or minimum value infinitely many times.

Counter-example.

The function below takes its absolute maximum value (=3) and its absolute minimum value (=1) an infinite number of times on the interval [1,4].

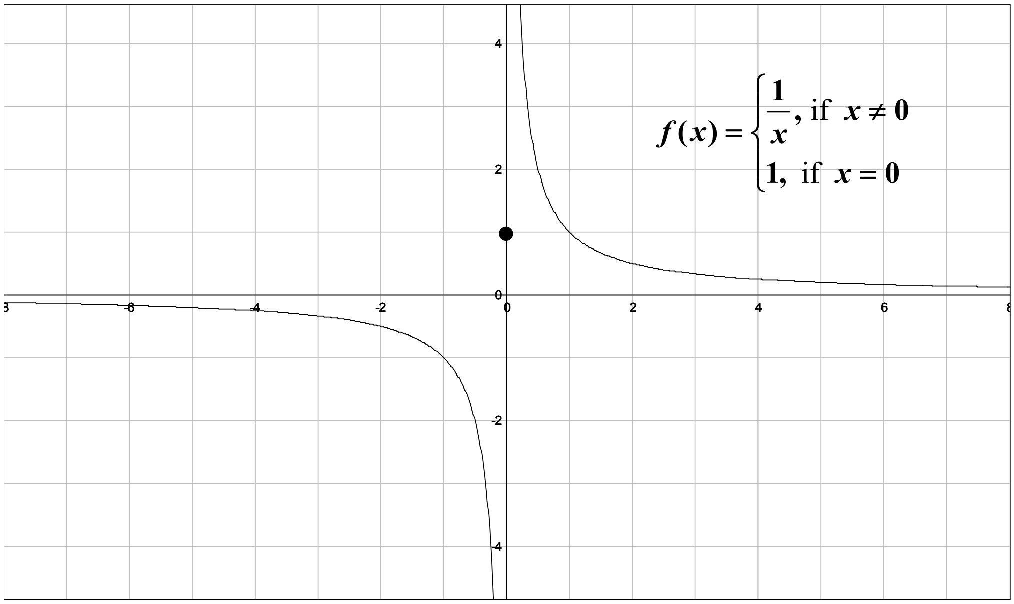


13. If a function $y = f(x)$ is defined on $[a,b]$ and $f(a)\times f(b)<0$ then there is some point $c\in(a,b)$ such that $f(c) = 0$.

Counter-example.

The function $f(x)=\begin{cases}\frac{1}{x}, \text{ if } x\neq 0 \\ 1, \text{ if } x=0\end{cases}$ is defined on [-1,1] and

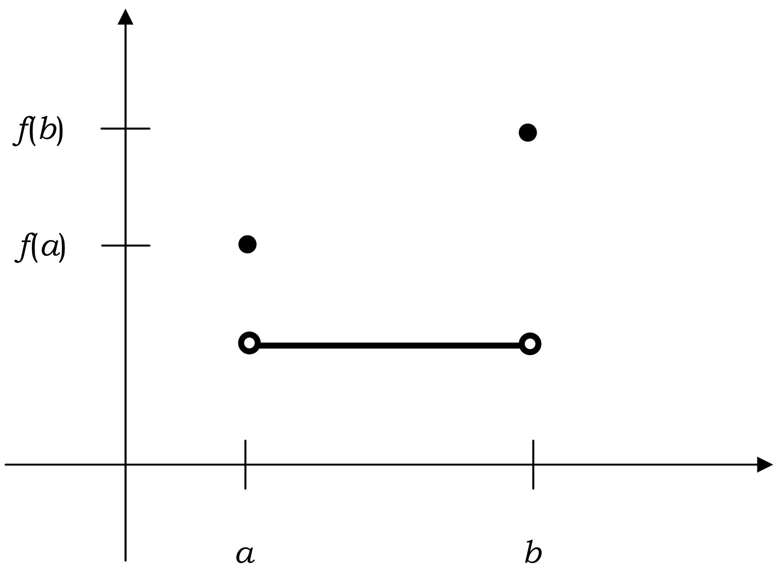
$f(-1) \times f(1) = (-1) \times (1) = -1 < 0$ but there is no point c on $[-1,1]$ such that $f(c) = 0$.



14. If a function $y = f(x)$ is defined on $[a,b]$ and continuous on (a,b) then for any $N \in (f(a), f(b))$ there is some point $c \in (a,b)$ such that $f(c) = N$.

Counter-example.

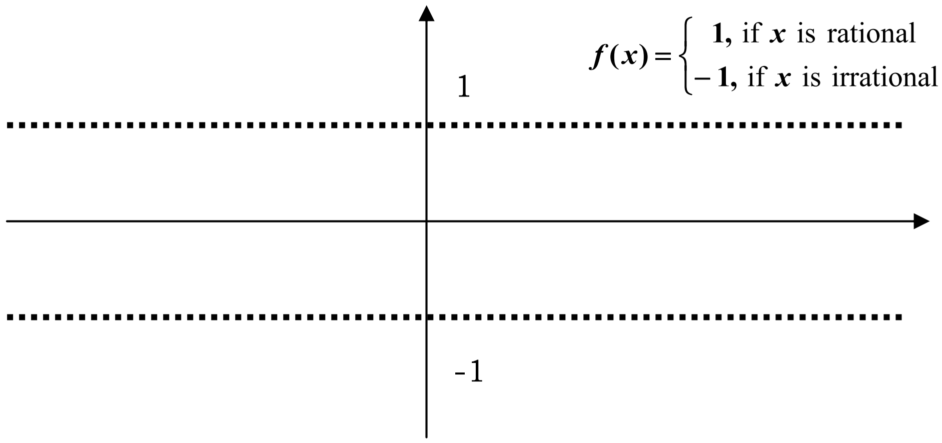
The function below is defined on $[a,b]$ and continuous on (a,b) but for any $N \in (f(a), f(b))$ there is no point $c \in (a,b)$ such that $f(c) = N$.



15. If a function is discontinuous at every point in its domain then the square and the absolute value of this function cannot be continuous.

Counter-example.

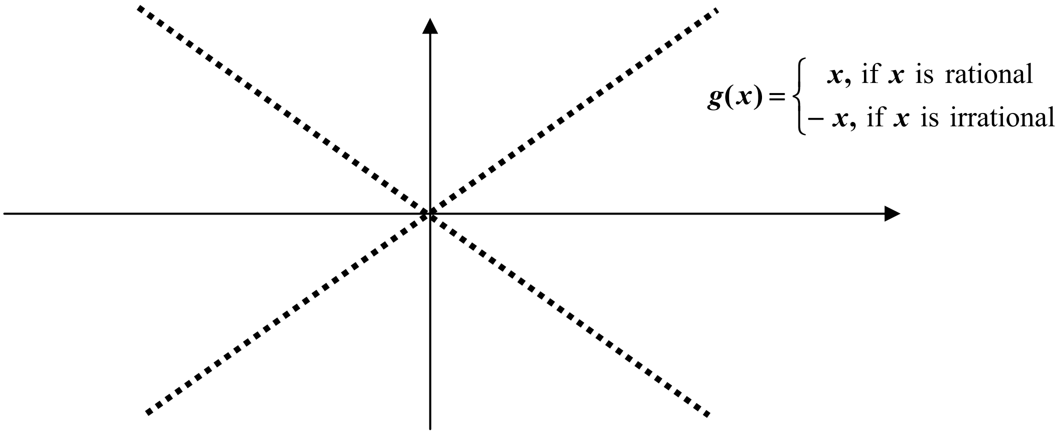
The function $f(x)=\begin{cases} 1, \text{ if } x \text{ is rational} \\ -1, \text{ if } x \text{ is irrational} \end{cases}$ is discontinuous at every point in its domain but both the square and the absolute value $f^2(x)=|f(x)|=1$ are continuous. It is impossible to draw the graph of the function $y = f(x)$ but the sketch below gives an idea of its behaviour.



16. A function cannot be continuous at only one point in its domain and discontinuous everywhere else.

Counter-example.

The function $g(x)=\begin{cases} x, \text{ if } x \text{ is rational} \\ -x, \text{ if } x \text{ is irrational} \end{cases}$ is continuous at the point $x = 0$ and discontinuous at all other points on R. It is impossible to draw the graph of the function $y = g(x)$ but the sketch below gives an idea of its behaviour.



17. A sequence of continuous functions on $[a,b]$ always converges to a continuous function on $[a,b]$.

<u>Counter-example</u>.

The sequence of continuous functions $f_n(x)=x^n$, $n\in N$ on $[0,1]$ converges to a discontinuous function when $n\to\infty$:

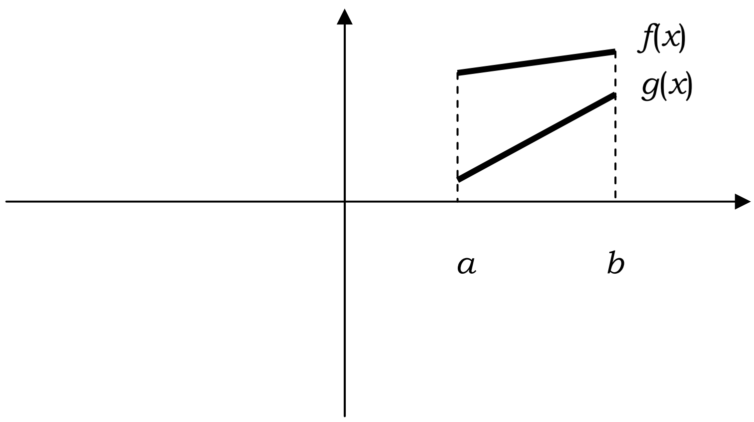
$$\lim_{n\to\infty} f_n(x)=\begin{cases} 0, \text{ if } x\in[0,1) \\ 1, \text{ if } x=1. \end{cases}$$

4. Differential Calculus

1. If both functions $y = f(x)$ and $y = g(x)$ are differentiable and $f(x) > g(x)$ on the interval (a,b) then $f'(x) > g'(x)$ on (a,b).

<u>Counter-example</u>.

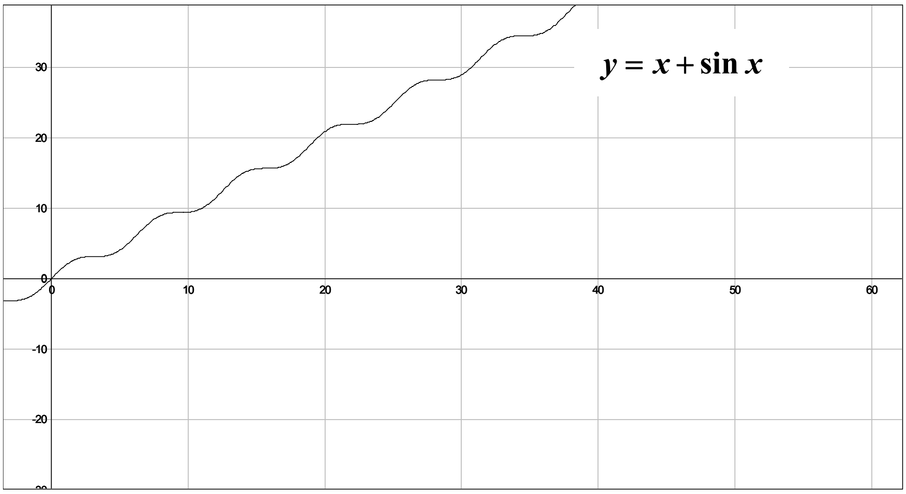
Both functions $y = f(x)$ and $y = g(x)$ are differentiable and $f(x) > g(x)$ on the interval (a,b) but $f'(x) < g'(x)$ on (a,b).

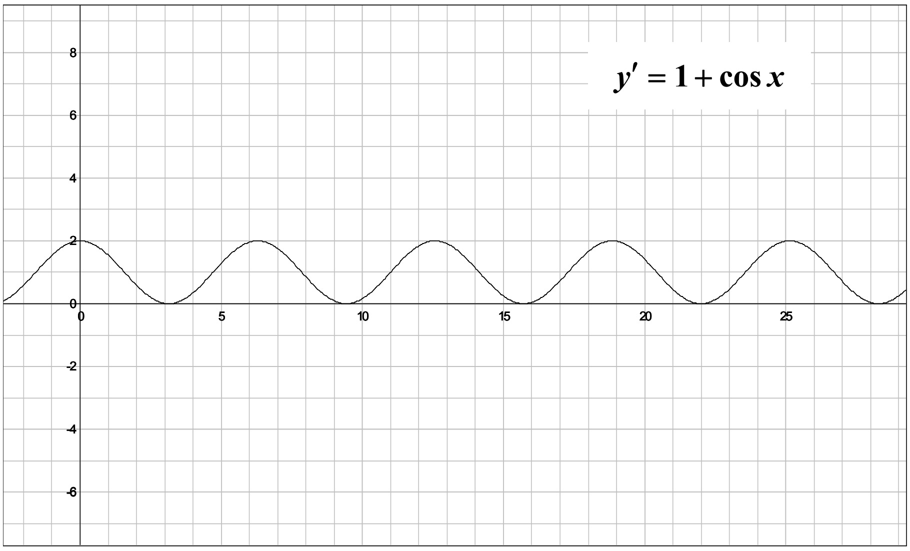


2. If a non-linear function is differentiable and monotone on $(0,\infty)$ then its derivative is also monotone on $(0,\infty)$.

<u>Counter-example</u>.

The non-linear function $y = x + \sin x$ is differentiable and monotone on $(0,\infty)$ but its derivative $y' = 1 + \cos x$ is not monotone on $(0,\infty)$.

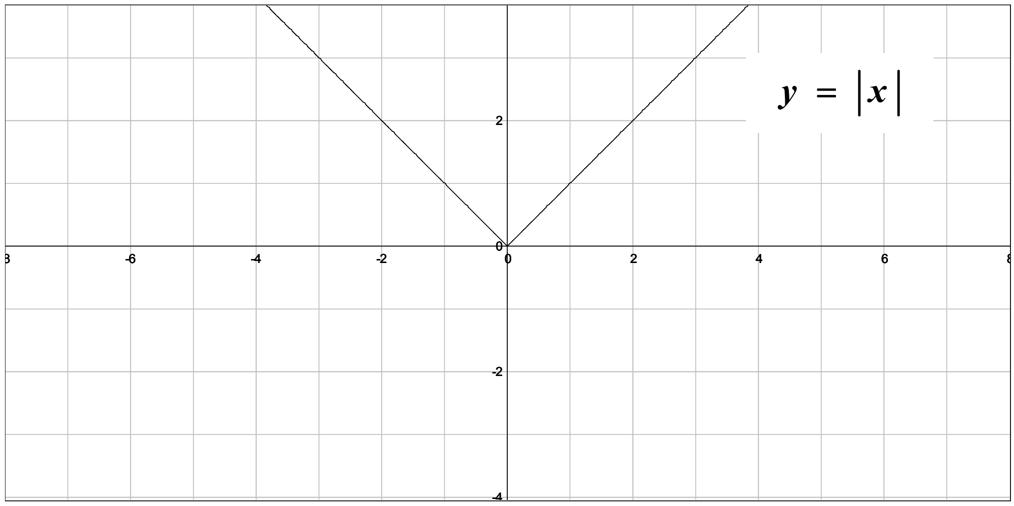




3. If a function is continuous at a point then it is differentiable at that point.

   Counter-example.

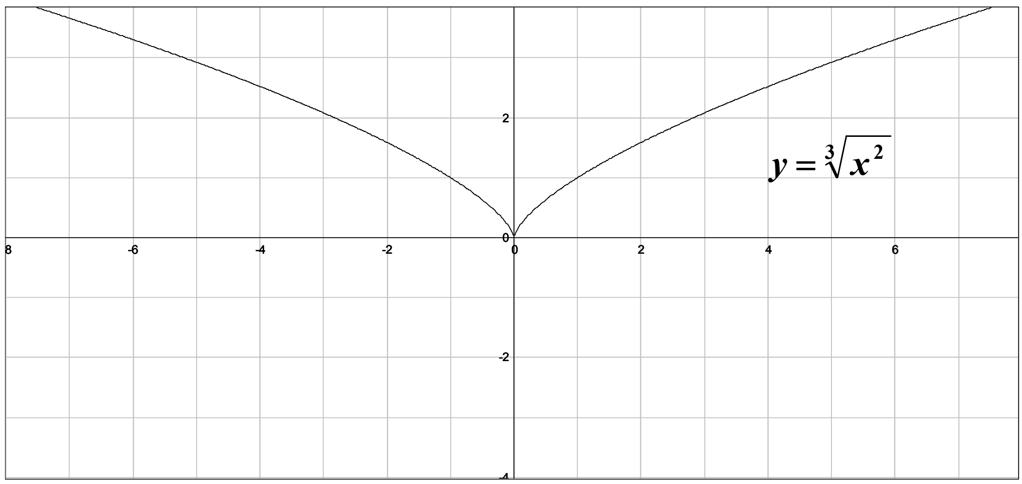
   The function $y = |x|$ is continuous at the point x = 0 but it is not differentiable at that point.



4. If a function is continuous on R and the tangent line exists at any point on its graph then the function is differentiable at any point on R.

Counter-example.

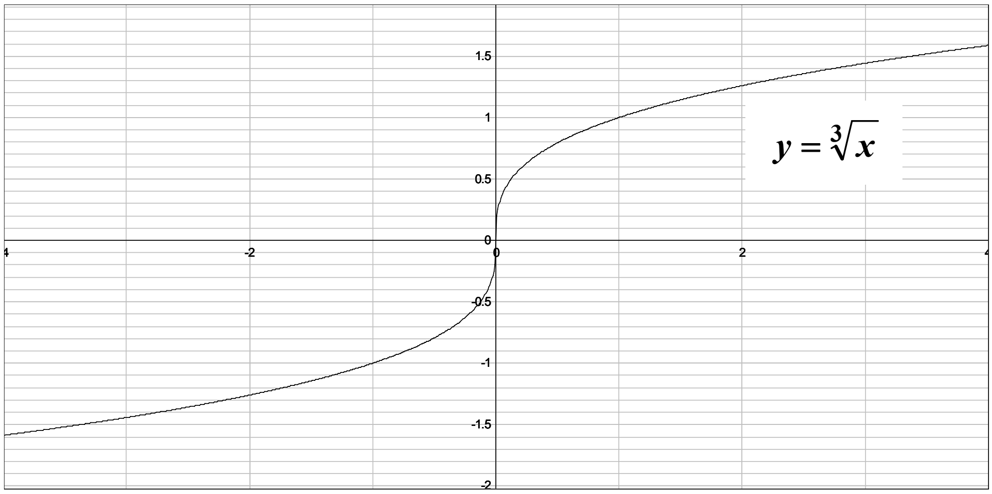
The function $y=\sqrt[3]{x^2}$ is continuous on R and the tangent line exists at any point on its graph but the function is not differentiable at the point $x = 0$.



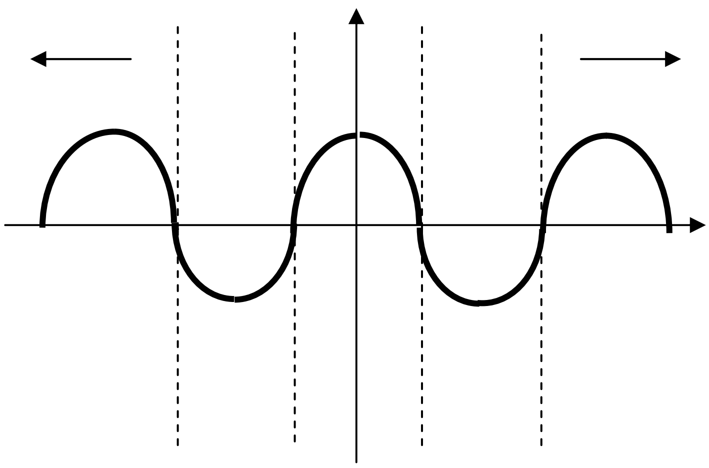
5. If a function is continuous on the interval (a,b) and its graph is a *smooth* curve (no sharp corners) on that interval then the function is differentiable at any point on (a,b).

Counter-example.

a) The function $y = \sqrt[3]{x}$ is continuous on R and its graph is a smooth curve (no sharp corners), but it is not differentiable at the point $x = 0$.



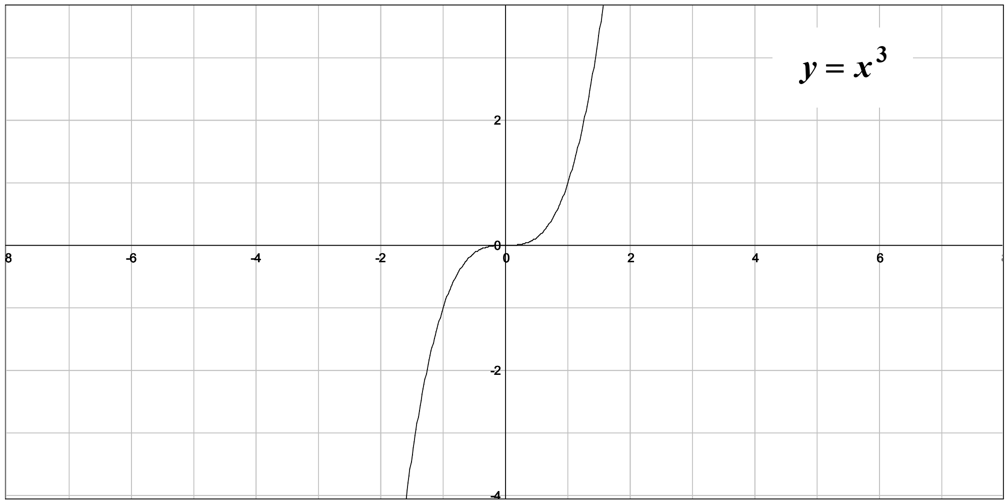
b) The function below is continuous on R and its graph is a smooth curve (no sharp corners), but it is non-differentiable at infinitely many points on R.



6. If the derivative of a function is zero at a point then the function is neither increasing nor decreasing at this point.

Counter-example.

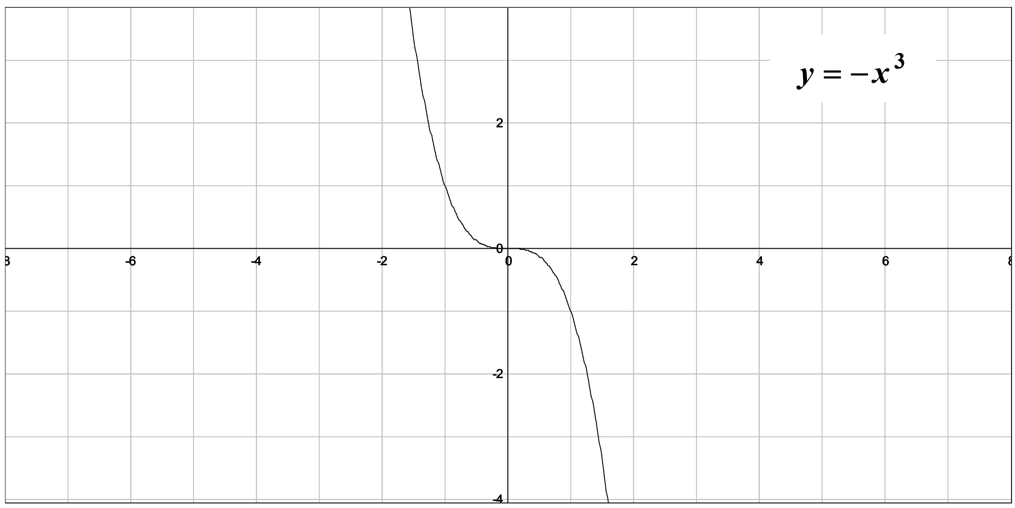
The derivative of the function $y = x^3$ is zero at the point $x = 0$ but the function is increasing at this point.



7. If a function is differentiable and decreasing on (a,b) then its gradient is negative on (a,b).

Counter-example.

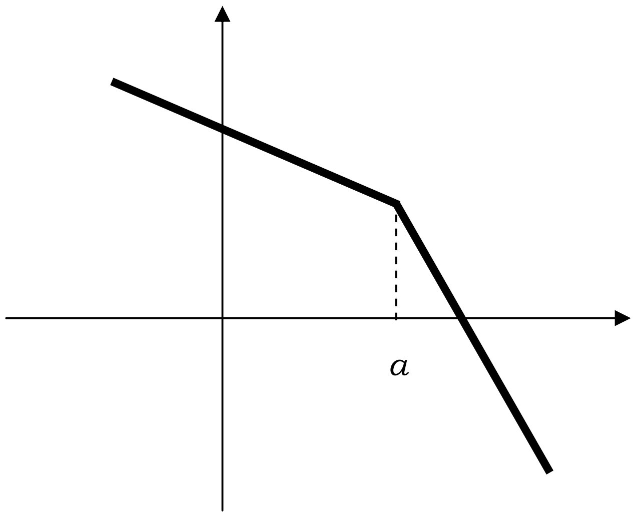
The function $y = -x^3$ is differentiable and decreasing on R but its gradient is zero at the point $x = 0$.



8. If a function is continuous and decreasing on (a,b) then its gradient is non-positive on (a,b).

Counter-example.

The function below is continuous and decreasing on R but its gradient doesn't exist at the point $x = a$.

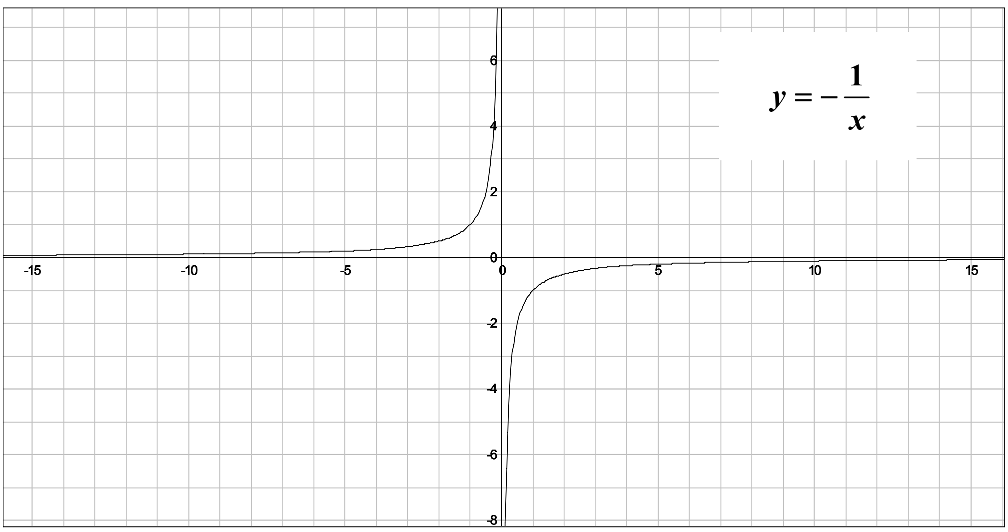


9. If a function has a positive derivative at any point in its domain then the function is increasing everywhere in its domain.

<u>Counter-example</u>.

The derivative of the function $y = -\frac{1}{x}$ $(x \neq 0)$ is $y' = \frac{1}{x^2}$, which is positive for all $x \neq 0$.

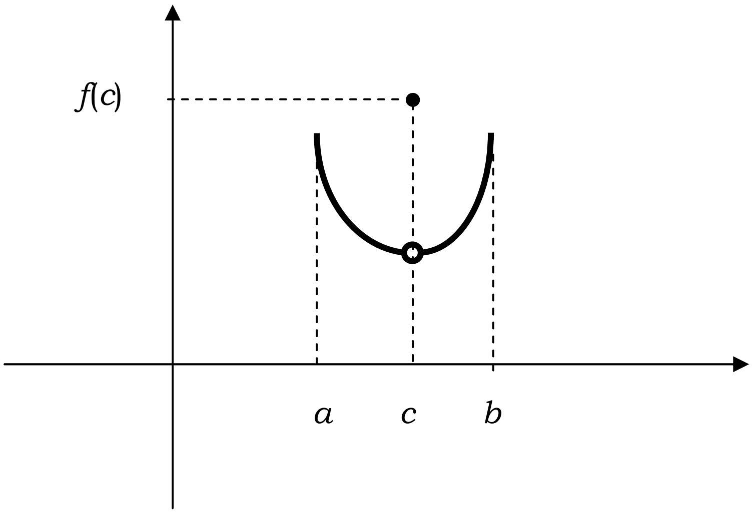
According to the definition, a function is increasing in its domain if for any x_1, x_2 from its domain from $x_1 < x_2$ it follows that $f(x_1) < f(x_2)$. If we take $x_1 = -1$ and $x_2 = 1$ $(x_1 < x_2)$ it follows that $f(x_1) > f(x_2)$.



10. If a function $y = f(x)$ is defined on $[a,b]$ and has a local maximum at the point $c \in (a,b)$ then in a sufficiently small neighbourhood of the point $x = c$ the function is increasing on the left and decreasing on the right from $x = c$.

Counter-example.

The function below is defined on $[a,b]$ and has a maximum at the point $c \in (a,b)$ but it is neither increasing on the left nor decreasing on the right from the point $x = c$.

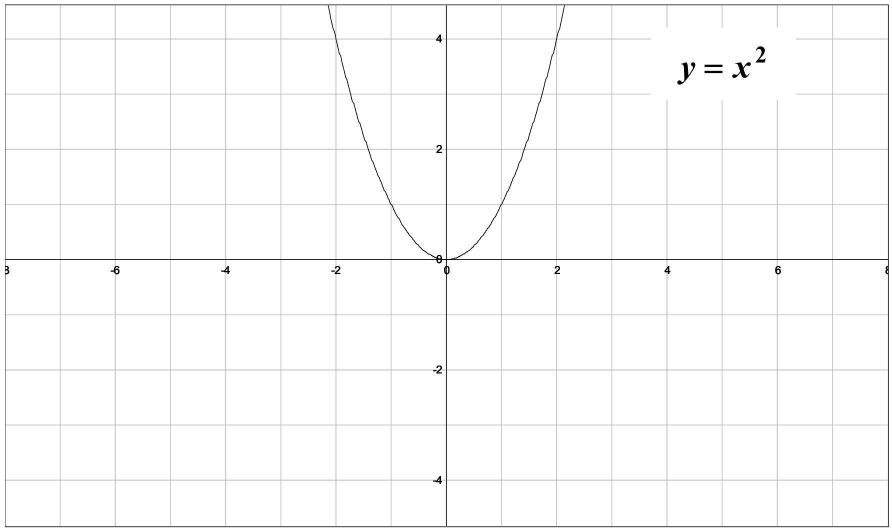


Comments. The definition of a local maximum requires neither differentiability nor continuity of a function at the point of interest: A function $y = f(x)$ has a local maximum at the point $x = c$ if $f(c) > f(x)$ for all x within a certain neighbourhood $(c-\delta, c+\delta), \delta > 0$ of the point $x = c$.

11. If a function $y = f(x)$ is differentiable for all real x and $f(0) = f'(0) = 0$ then $f(x) = 0$ for all real x.

Counter-example.

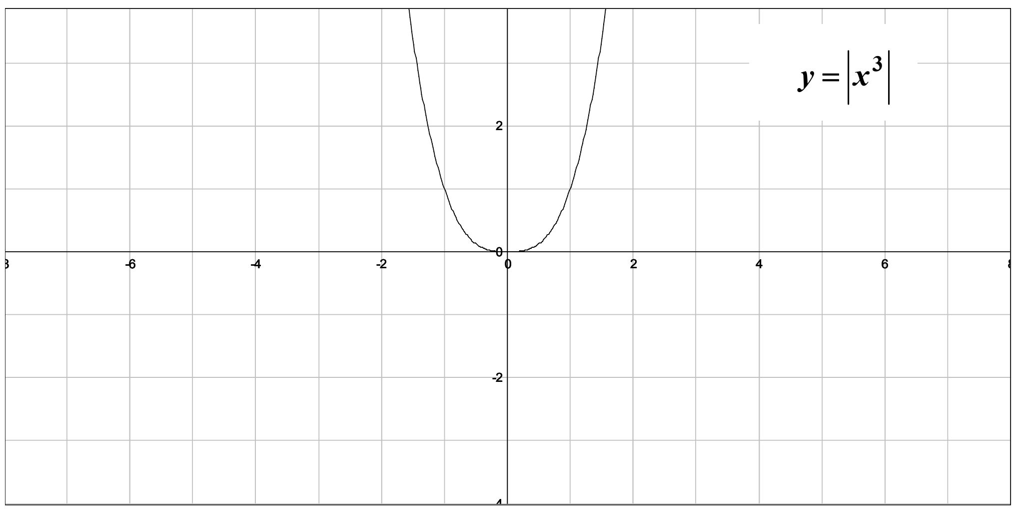
Both the function $y = x^2$ and its derivative $y' = 2x$ equal zero at the point $x = 0$ but the function is not zero for all real x.



12. If a function $y = f(x)$ is differentiable on the interval (a,b) and takes both positive and negative values on (a,b) then its absolute value $|f(x)|$ is not differentiable at the point(s) where $f(x) = 0$, e.g. $|f(x)| = |x|$ or $|f(x)| = |\sin x|$.

<u>Counter-example</u>.

The function $y = x^3$ is differentiable on R and takes both positive and negative values but its absolute value $|y| = |x^3|$ is differentiable at the point $x = 0$ where the function equals zero.

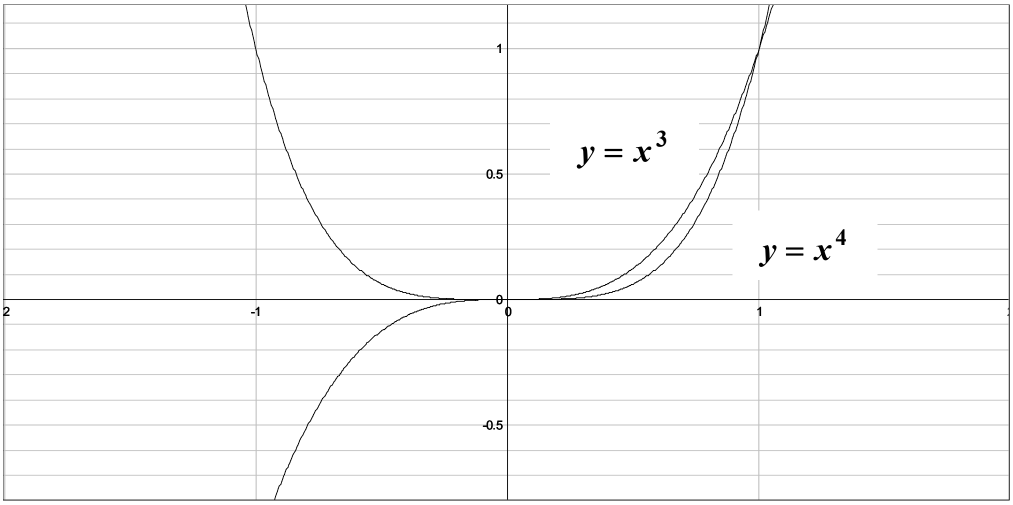


<u>Comments</u>. To make the statement true it should conclude: "…then its absolute value $|f(x)|$ is not differentiable at the points where $f(x) = 0$ and $f'(x) \neq 0$."

13. If both functions $y = f(x)$ and $y = g(x)$ are differentiable on the interval (a,b) and intersect somewhere on (a,b) then the function $\max\{f(x), g(x)\}$ is not differentiable at the point(s) where $f(x) = g(x)$.

<u>Counter-example</u>.

The function $\max\{x^3, x^4\}$ on (-1,1) is differentiable at the point $x = 0$ where the functions $y = x^3$ and $y = x^4$ intersect.

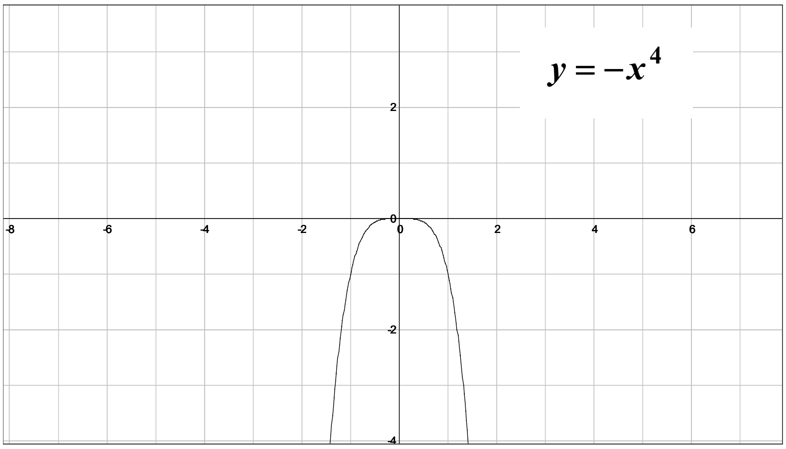


<u>Comments</u>. To make the statement true it should conclude: "...then the function $\max\{f(x), g(x)\}$ is not differentiable at the point(s) where $f(x) = g(x)$ and $f'(x) \neq g'(x)$."

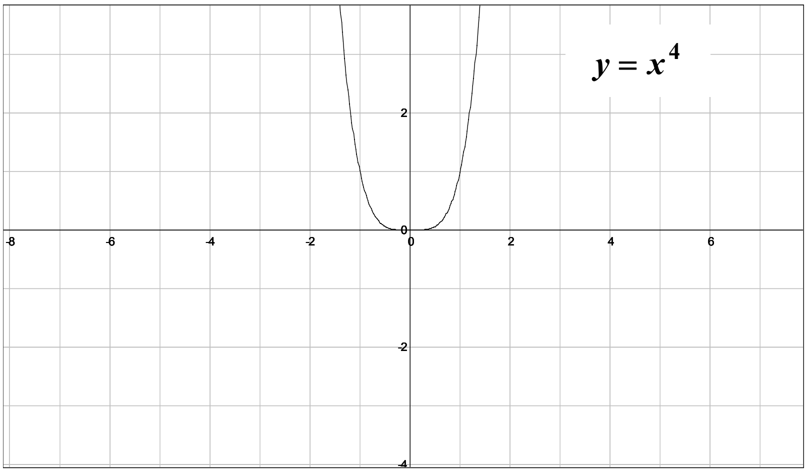
14. If a function is twice differentiable at a local maximum (minimum) point then its second derivative is negative (positive) at that point.

<u>Counter-example</u>.

The function $y = -x^4$ is twice differentiable at its maximum point $x = 0$ but the second derivative is zero at this point.



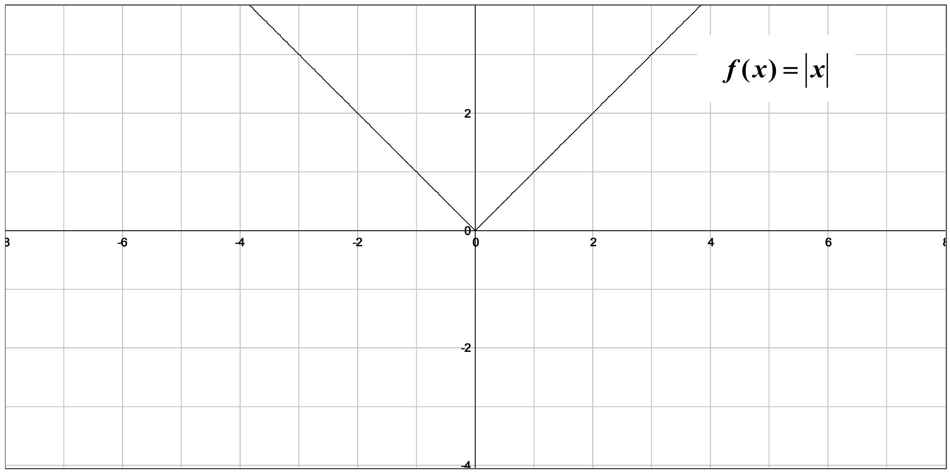
The function $y = x^4$ is twice differentiable at its minimum point $x = 0$ but the second derivative is zero at that point:

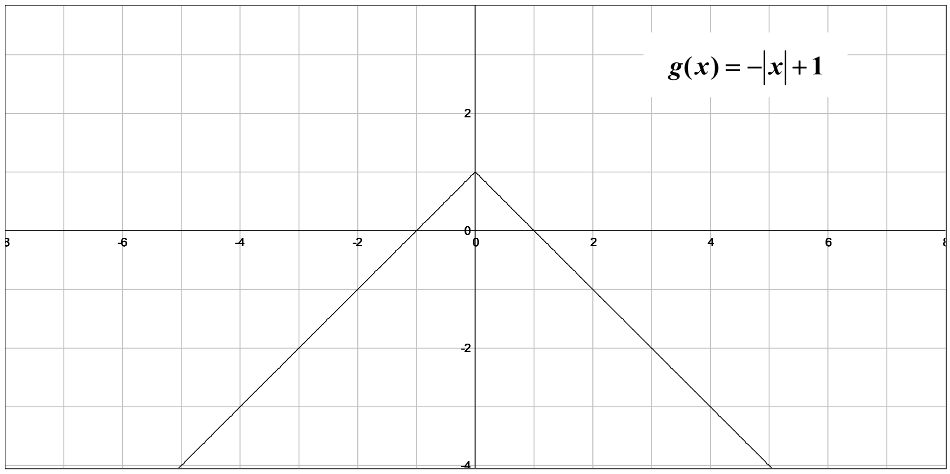


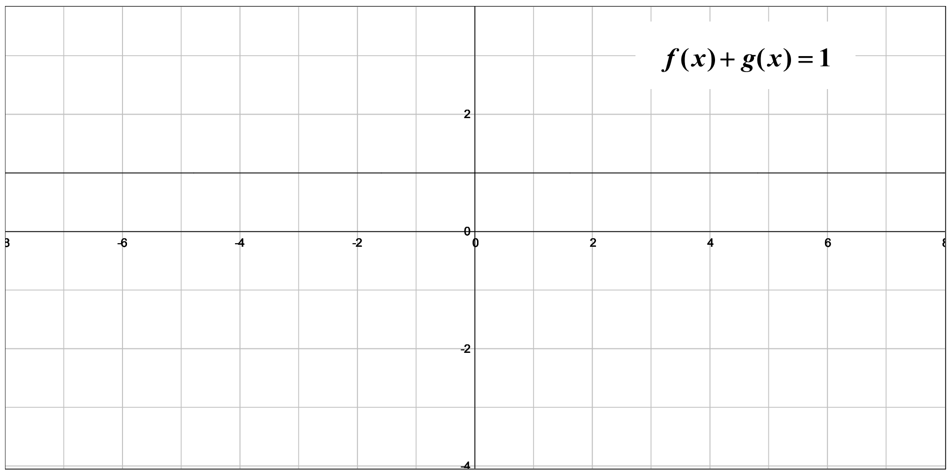
15. If both functions $y = f(x)$ and $y = g(x)$ are non-differentiable at $x = a$ then $f(x) + g(x)$ is also not differentiable at $x = a$.

<u>Counter-example</u>.

Both functions $f(x) = |x|$ and $g(x) = -|x| + 1$ are not differentiable at $x = 0$ but $f(x) + g(x) = 1$ is differentiable at any x including $x = 0$.





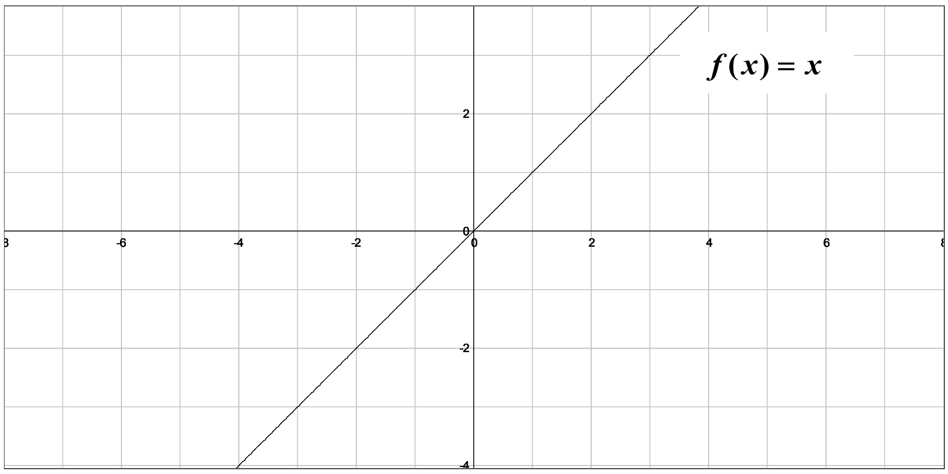


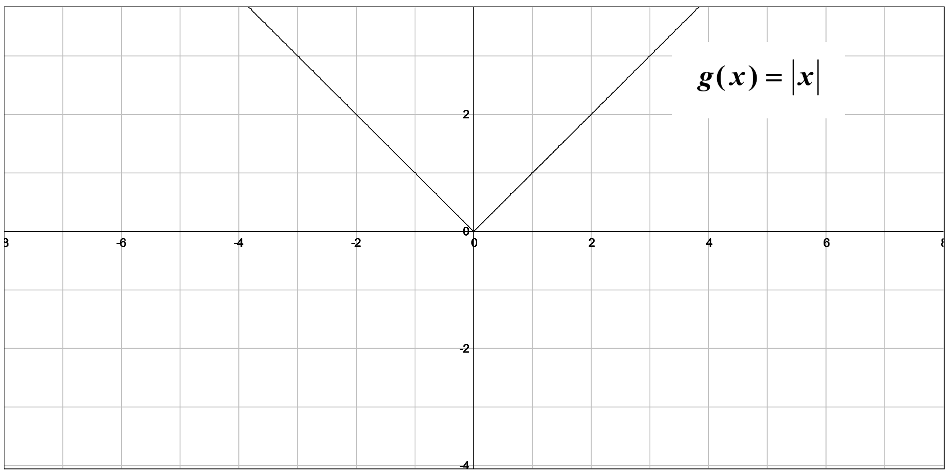
Comments. More generally, $f(x) = A(x)$ and $g(x) = B(x) - A(x)$, where $A(x)$ is not differentiable and $B(x)$ is differentiable at $x = a$. Both $f(x)$ and $g(x)$ are not differentiable, but $f(x) + g(x) = B(x)$ is differentiable at $x = a$.

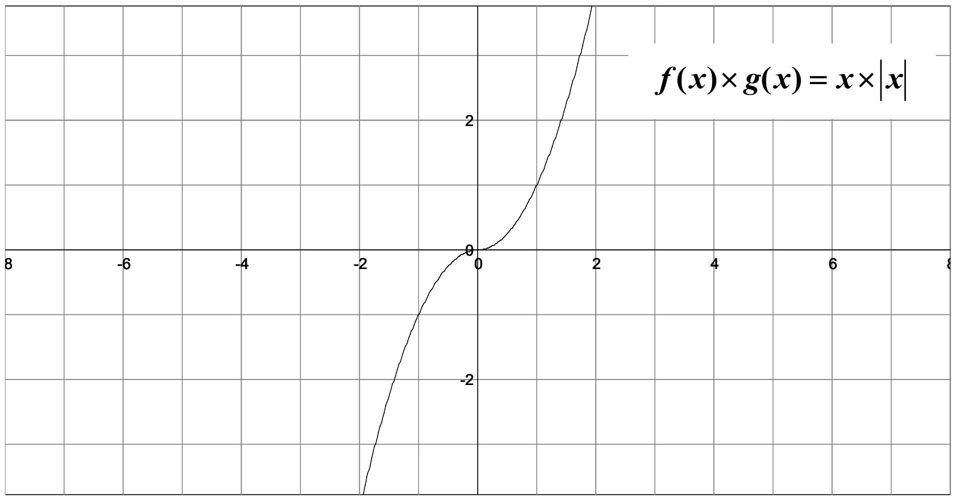
16. If a function $y = f(x)$ is differentiable and a function $y = g(x)$ is not differentiable at $x = a$ then $f(x) \times g(x)$ is not differentiable at $x = a$.

<u>Counter-example</u>.

The function $f(x) = x$ is differentiable at $x = 0$ and the function $g(x) = |x|$ is not differentiable at $x = 0$, but the function $f(x) \times g(x) = x \times |x|$ is differentiable at the point $x = 0$ (the derivative equals zero).



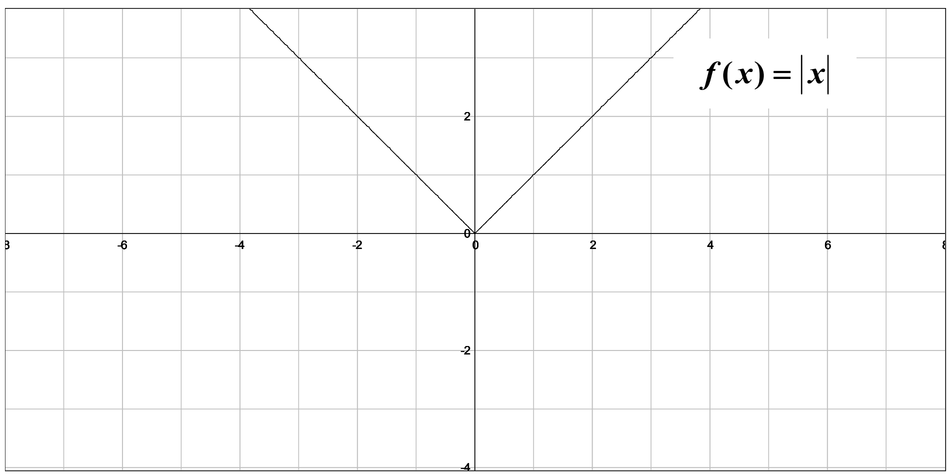


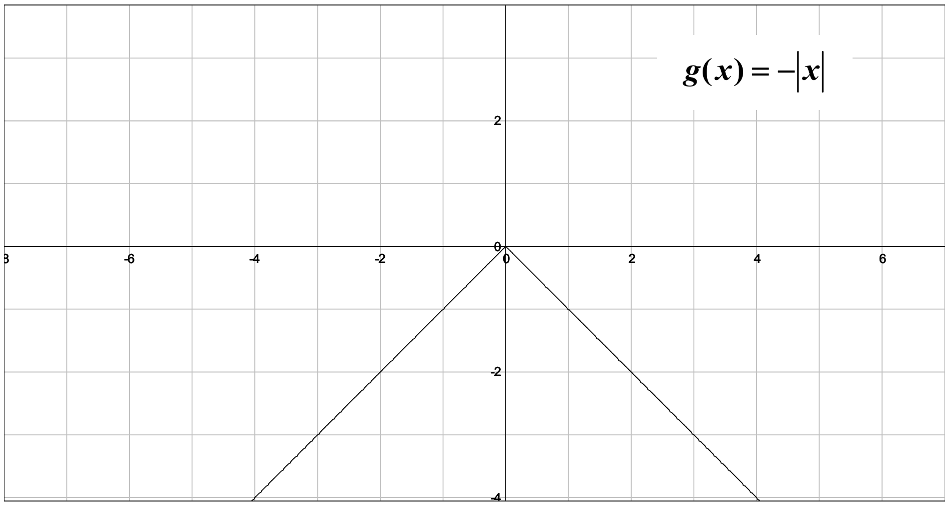


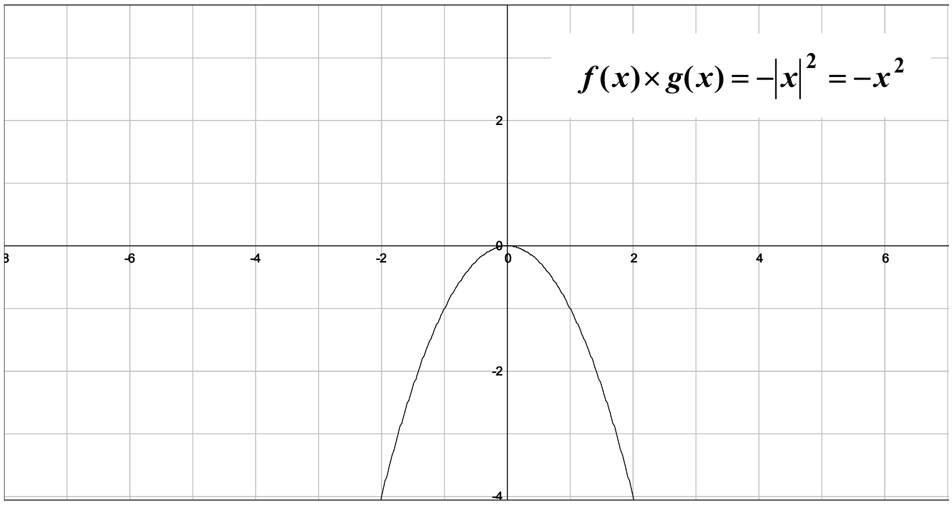
17. If both functions $y = f(x)$ and $y = g(x)$ are not differentiable at $x = a$ then $f(x)\times g(x)$ is also not differentiable at $x = a$.

<u>Counter-example</u>.

Both functions $f(x)=|x|$ and $g(x)=-|x|$ are not differentiable at the point $x = 0$ but the function $f(x)\times g(x)=-|x|^2=-x^2$ is differentiable at $x = 0$.







18. If a function $y = g(x)$ is differentiable at $x = a$ and a function $y = f(x)$ is not differentiable at $g(a)$ then the function $F(x) = f(g(x))$ is not differentiable at $x = a$.

<u>Counter-example</u>.

The function $g(x) = x^2$ is differentiable at $x = 0$, and the function $f(x) = |x|$ is not differentiable at $g(0)=0$, but the function $F(x) = f(g(x)) = |x^2| = x^2$ is differentiable at $x = 0$.

19. If a function $y = g(x)$ is not differentiable at $x = a$ and a function $y = f(x)$ is differentiable at $g(a)$ then the function $F(x) = f(g(x))$ is not differentiable at $x = a$.

Counter-example.

The function $g(x)=|x|$ is not differentiable at $x = 0$, the function $f(x)=x^2$ is differentiable at $g(0) = 0$, but the function $F(x)=f(g(x))=|x|^2=x^2$ is differentiable at $x = 0$.

20. If a function $y = g(x)$ is not differentiable at $x = a$ and a function $y = f(x)$ is not differentiable at $g(a)$ then the function $F(x) = f(g(x))$ is not differentiable at $x = a$.

Counter-example.

The function $g(x)=\frac{2}{3}x-\frac{1}{3}|x|$ is not differentiable at $x = 0$ and the function $f(x)=2x+|x|$ is not differentiable at $g(0) = 0$, but the function $F(x)=f(g(x))=2(\frac{2}{3}x-\frac{1}{3}|x|)+\left|\frac{2}{3}x-\frac{1}{3}|x|\right|$ is differentiable at $x = 0$.

Let us show this using the definition of the derivative.

$$F'(0)=\lim_{\Delta x\to 0}\frac{F(\Delta x)-F(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{2(\frac{2}{3}\Delta x-\frac{1}{3}|\Delta x|)+\left|\frac{2}{3}\Delta x-\frac{1}{3}|\Delta x|\right|}{\Delta x}$$

If $\Delta x\to 0^-$ then $\lim\limits_{\Delta x\to 0^-}\dfrac{2(\frac{2}{3}\Delta x+\frac{1}{3}\Delta x)+\left|\frac{2}{3}\Delta x+\frac{1}{3}\Delta x\right|}{\Delta x}=\lim\limits_{\Delta x\to 0^-}\dfrac{2\Delta x-\Delta x}{\Delta x}=1$

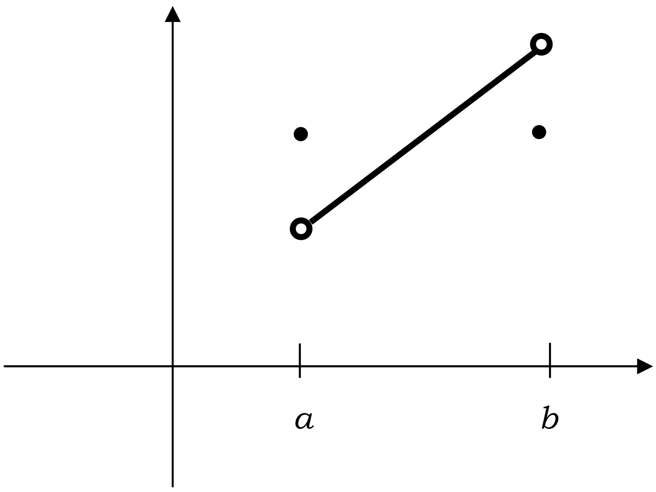
If $\Delta x\to 0^+$ then $\lim\limits_{\Delta x\to 0^+}\dfrac{2(\frac{2}{3}\Delta x-\frac{1}{3}\Delta x)+\left|\frac{2}{3}\Delta x-\frac{1}{3}\Delta x\right|}{\Delta x}=\lim\limits_{\Delta x\to 0^+}\dfrac{\frac{2}{3}\Delta x+\frac{1}{3}\Delta x}{\Delta x}=1$

Therefore $F'(0)=1.$ (Another way is to show that $F(x) = x$).

21. If a function $y = f(x)$ is defined on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$, then there exists a point $c\in(a,b)$ such that $f'(c)=0$.

Counter-example.

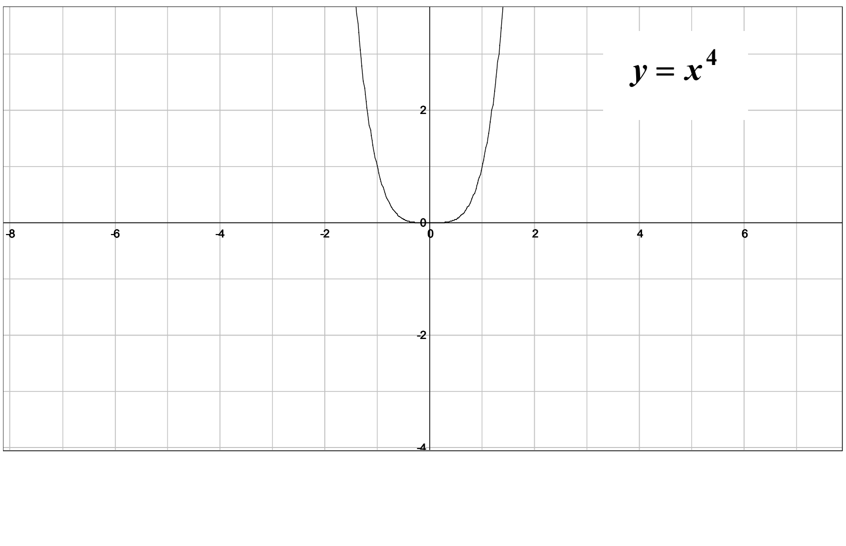
The function below is defined on $[a,b]$, differentiable on (a,b) and $f(a) = f(b)$ but there is no such a point $c \in (a,b)$ that $f'(c) = 0$.



22. If a function is twice-differentiable in a certain neighbourhood around $x = a$ and its second derivative is zero at that point then the point $(a,f(a))$ is a point of inflection for the graph of the function.

Counter-example.

The function $y = x^4$ is twice differentiable on R and its second derivative is zero at the point $x = 0$ but the point (0,0) is not a point of inflection.

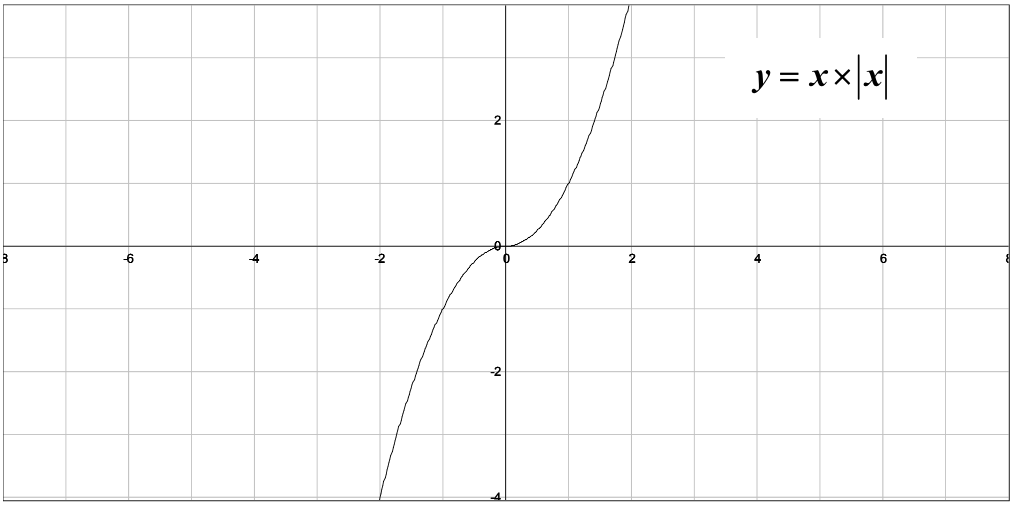


23. If a function $y = f(x)$ is differentiable at the point $x = a$ and the point $(a, f(a))$ is a point of inflection on the function's graph then the second derivative is zero at that point.

<u>Counter-example</u>.

The function $y = x \times |x|$ is differentiable at $x = 0$ and the point (0,0) is a point of inflection but the second derivative does not exist at

$x = 0$.



24. If both functions $y = f(x)$ and $y = g(x)$ are differentiable on R then to evaluate the limit $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}$ in the indeterminate form of type $\left[\frac{\infty}{\infty}\right]$ we can use the following rule: $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}$

<u>Counter-example</u>.

If we use the above "rule" to find the limit $\lim\limits_{x\to\infty}\frac{6x+\sin x}{2x+\sin x}$ then:

$$\lim_{x\to\infty}\frac{6x+\sin x}{2x+\sin x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty}\frac{6+\cos x}{2+\cos x} \text{ is undefined.}$$

But the limit $\lim\limits_{x\to\infty}\frac{6x+\sin x}{2x+\sin x}$ exists and equals 3:

$$\lim_{x\to\infty}\frac{6x+\sin x}{2x+\sin x} = \lim_{x\to\infty}\frac{6+\frac{\sin x}{x}}{2+\frac{\sin x}{x}} = 3.$$

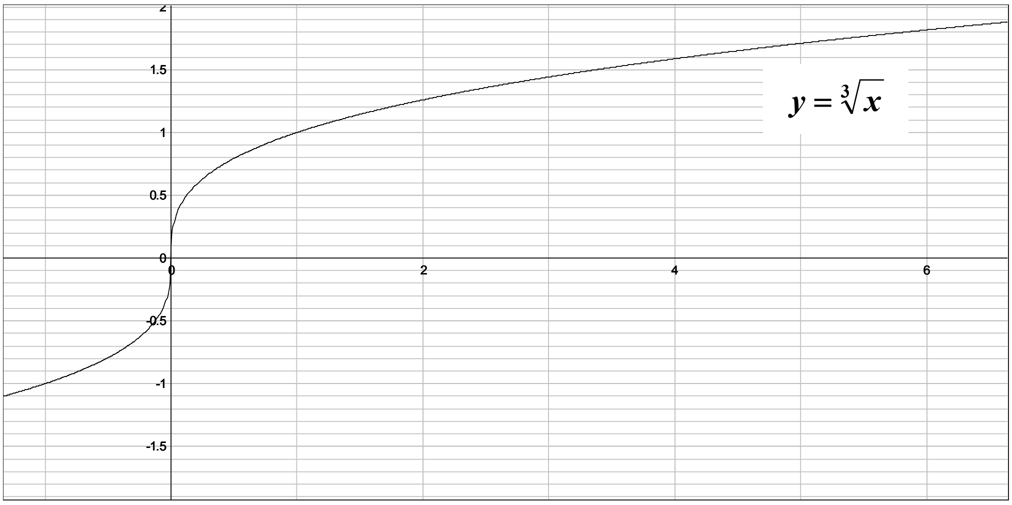
Comments. To make the above 'rule' correct we need to add "if the limit $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$ exists or equals $\pm\infty$". This is the well-known l'Hopital's Rule for limits.

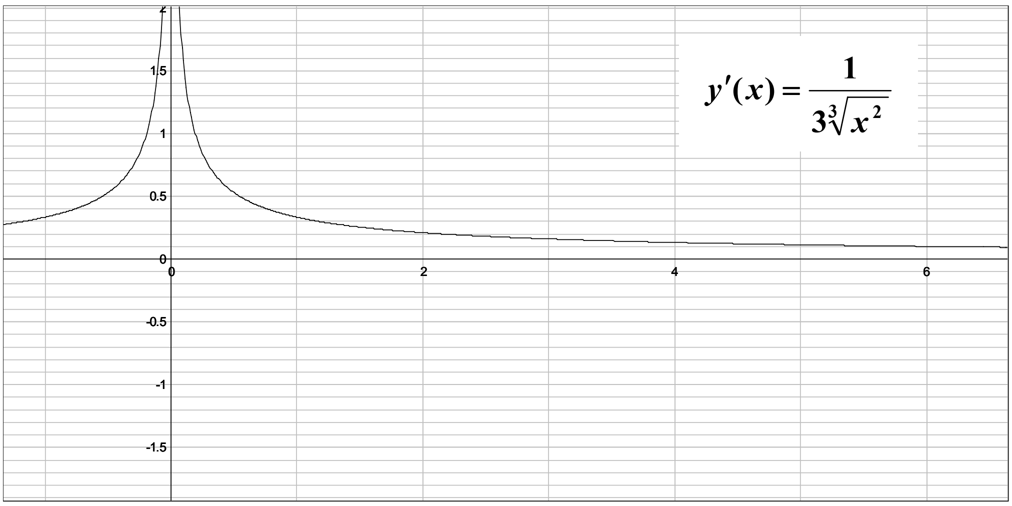
25. If a function $y = f(x)$ is differentiable on (a,b) and $\lim_{x\to a^+} f'(x) = \infty$ then $\lim_{x\to a^+} f(x) = \infty$.

Counter-example.

The function $y = \sqrt[3]{x}$ is differentiable on (0,1) and

$\lim_{x\to 0^+} y'(x) = \lim_{x\to 0^+}\frac{1}{3\sqrt[3]{x^2}} = \infty$ but $\lim_{x\to 0^+} y(x) = \lim_{x\to 0^+}\sqrt[3]{x} = 0$.

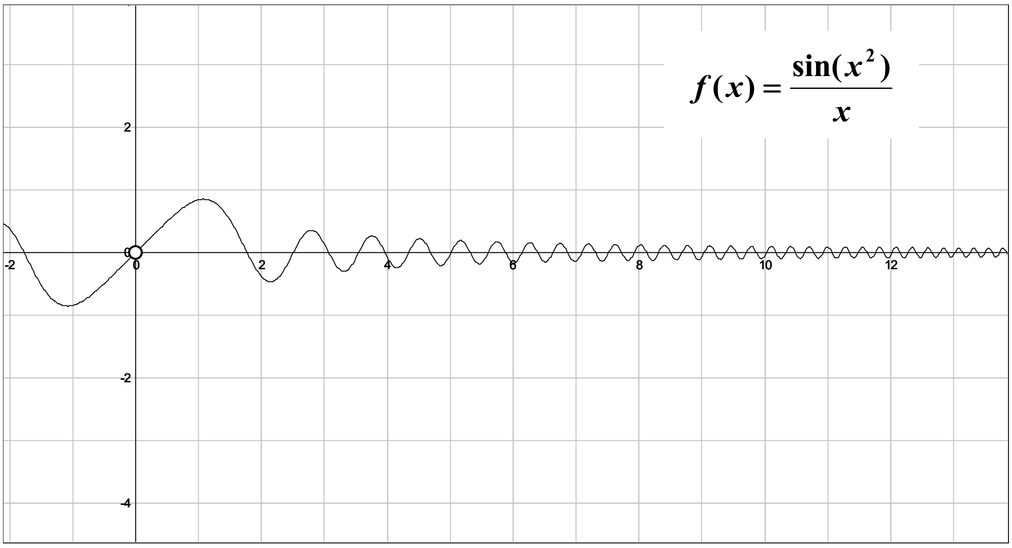


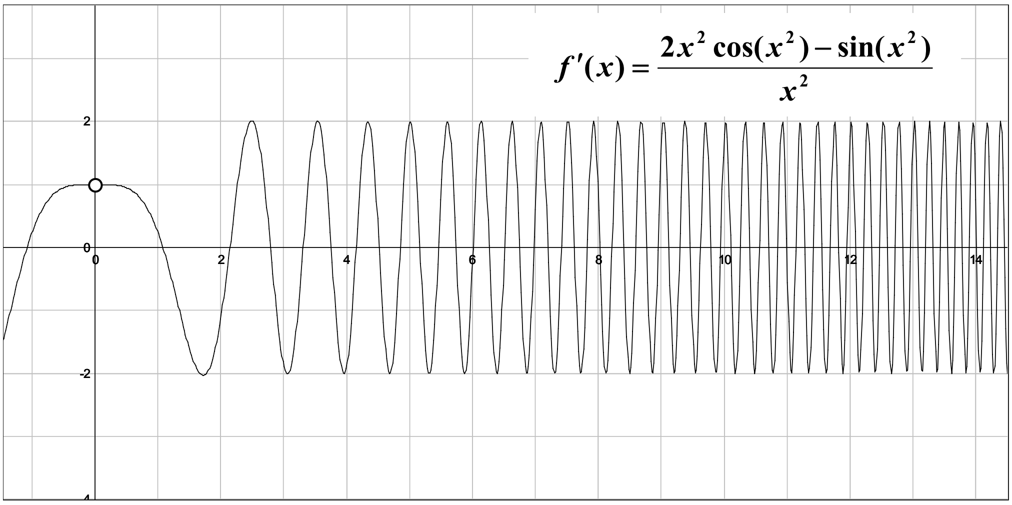


26. If a function $y = f(x)$ is differentiable on $(0,\infty)$ and $\lim_{x\to\infty} f(x)$ exists then $\lim_{x\to\infty} f'(x)$ also exists.

<u>Counter-example</u>.

The function $f(x)=\dfrac{\sin(x^2)}{x}$ is differentiable on $(0,\infty)$ and $\lim_{x\to\infty}\dfrac{\sin(x^2)}{x}=0$ but $\lim_{x\to\infty} f'(x)=\lim_{x\to\infty}\dfrac{2x^2\cos(x^2)-\sin(x^2)}{x^2}$ does not exist.

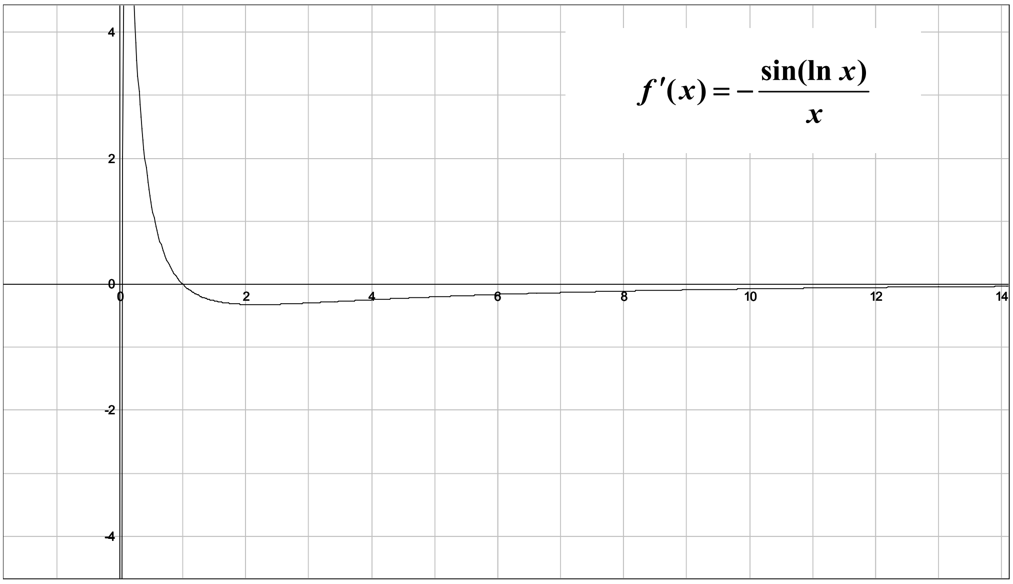


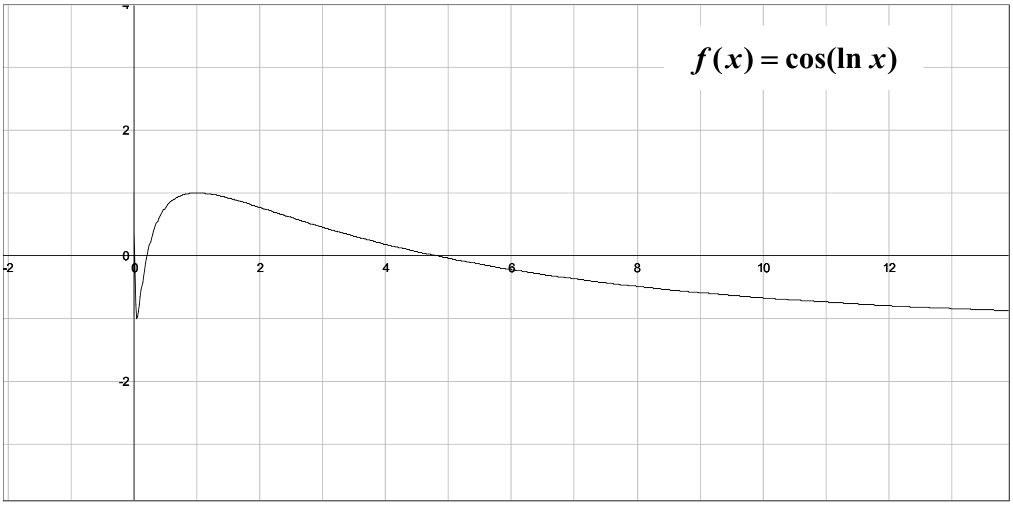


27. If a function $y = f(x)$ is differentiable and bounded on $(0,\infty)$ and $\lim_{x\to\infty} f'(x)$ exists then $\lim_{x\to\infty} f(x)$ also exists.

Counter-example.

The function $f(x) = \cos(\ln x)$ is differentiable and bounded on $(0.\infty)$ and the limit of its derivative exists: $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} -\frac{\sin(\ln x)}{x} = 0$. However, the limit of the function $\lim_{x\to\infty} \cos(\ln x)$ does not exist.



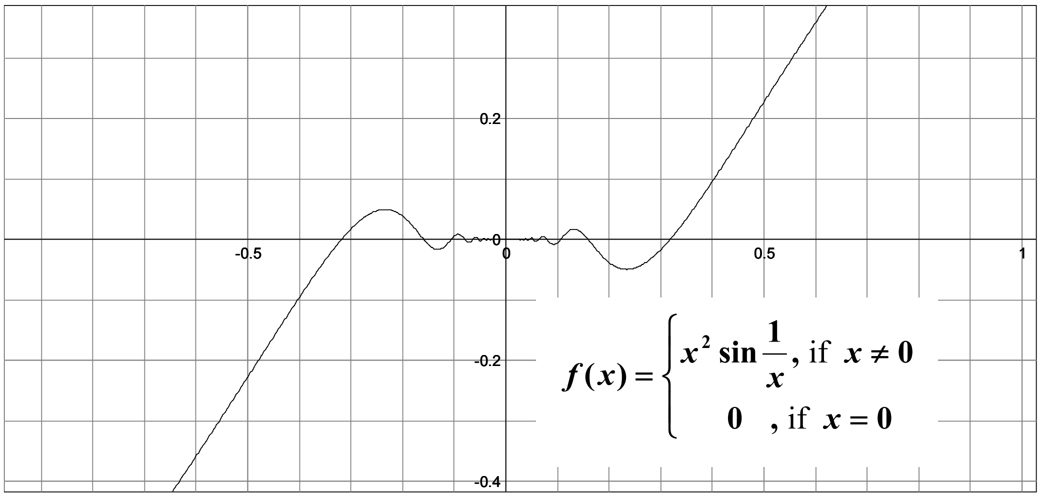


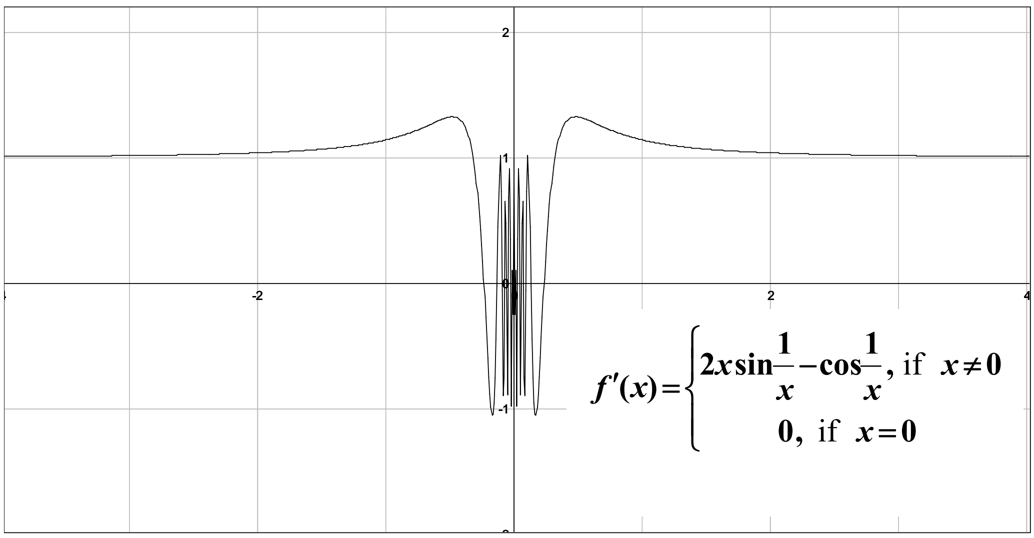
28. If a function $y = f(x)$ is differentiable at the point $x = a$ then its derivative is continuous at $x = a$.

Counter-example.

The function $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is differentiable at $x = 0$

but its derivative $f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x}, \text{ if } x \neq 0 \\ 0 \quad , \text{ if } x = 0 \end{cases}$ is discontinuous at $x = 0$.



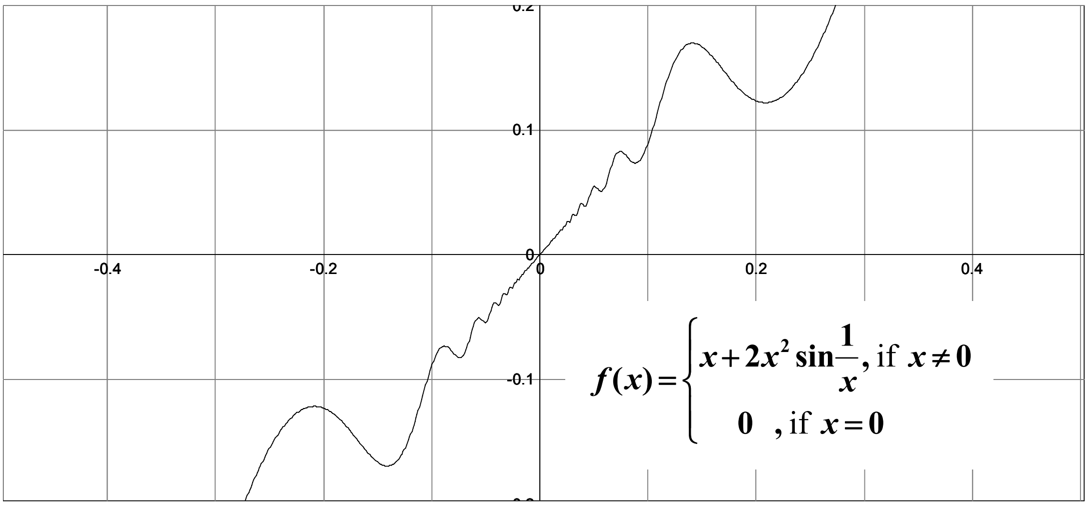


29. If the derivative of a function $y = f(x)$ is positive at the point $x = a$ then there is a neighbourhood about $x = a$ (no matter how small) where the function is increasing.

Counter-example.

The function $f(x) = \begin{cases} x + 2x^2 \sin\dfrac{1}{x}, \text{ if } x \neq 0 \\ 0 \ , \text{ if } x = 0 \end{cases}$ has the derivative

$$f'(x) = \begin{cases} 1 + 4x\sin\frac{1}{x} - 2\cos\frac{1}{x}, \text{ if } x \neq 0 \\ 1 \quad , \text{ if } x = 0 \end{cases}$$ which is positive at x = 0 but it takes positive and negative values in any neighbourhood of the point x = 0. This means the function $y = f(x)$ is not monotone in any neighbourhood of the point x = 0.

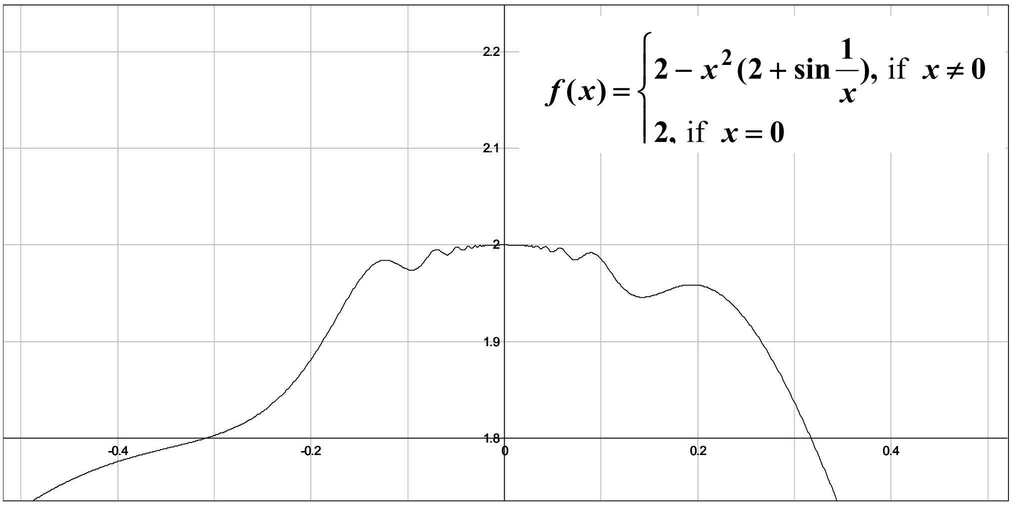


30. If a function $y = f(x)$ is continuous on (a,b) and has a local maximum at the point $c \in (a,b)$ then in a sufficiently small neighbourhood of the point $x = c$ the function is increasing on the left and decreasing on the right from $x = c$.

Counter-example.

The function $f(x) = \begin{cases} 2 - x^2(2 + \sin\frac{1}{x}), \text{ if } x \neq 0 \\ 2, \text{ if } x = 0 \end{cases}$ is continuous on R.

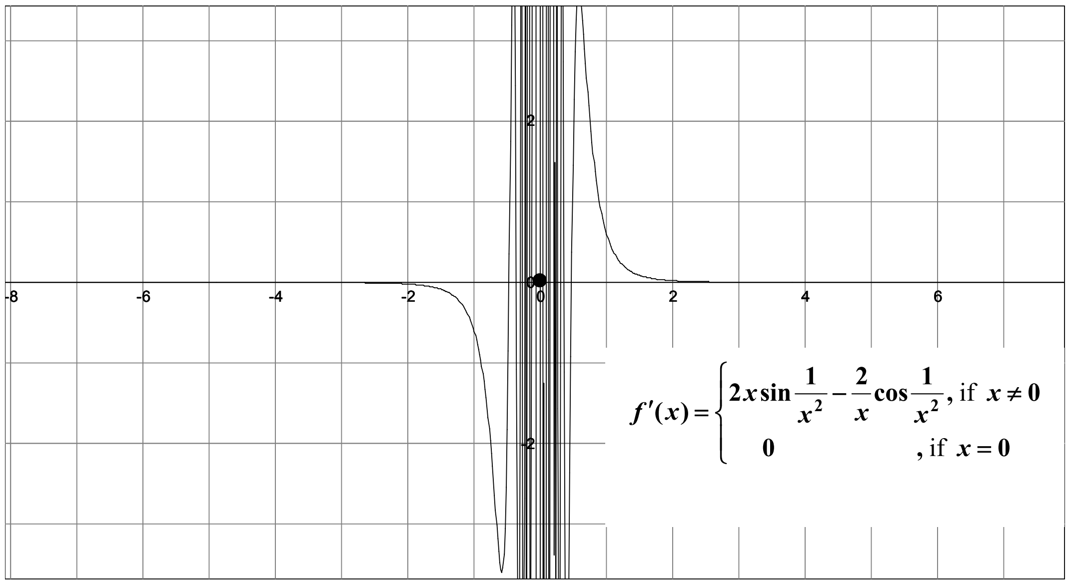
Since $x^2(2 + \sin\frac{1}{x})$ is positive for all $x \neq 0$ then $2 > 2 - x^2(2 + \sin\frac{1}{x})$. Therefore the function $y = f(x)$ has a local maximum at the point x = 0. But it is neither increasing on the left nor decreasing on the right in any neighbourhood of the point x = 0. To show this we can find the derivative $f'(x) = -4x - 2x\sin\frac{1}{x} + \cos\frac{1}{x}$; $x \neq 0$. The derivative takes both positive and negative values in any interval $(-\delta,0) \cup (0,\delta)$ and therefore the function is not monotone in any interval $(-\delta,0) \cup (0,\delta)$, where $\delta > 0$.



31. If a function $y = f(x)$ is differentiable at the point $x = a$ then there is a certain neighbourhood of the point $x = a$ where the derivative of the function $y = f(x)$ is bounded.

Counter-example.

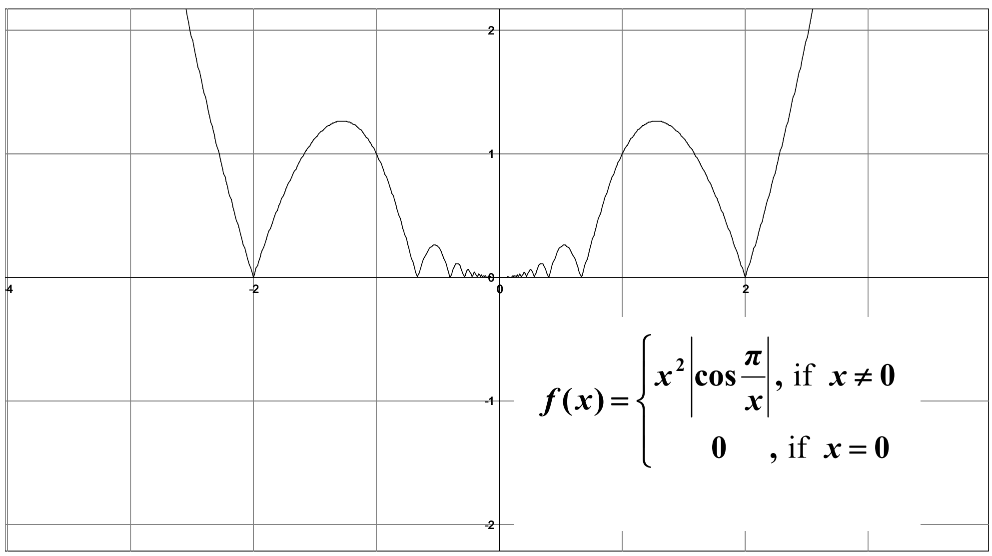
The function $f(x)=\begin{cases} x^2\sin\frac{1}{x^2}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$ is differentiable at the point $x = 0$. Its derivative is $f'(x)=\begin{cases} 2x\sin\frac{1}{x^2}-\frac{2}{x}\cos\frac{1}{x^2}, & \text{if } x\neq 0 \\ 0, & \text{if } x=0 \end{cases}$. The derivative of the function $y = f(x)$ is unbounded in any neighbourhood of the point $x = 0$.



32. If a function $y = f(x)$ at any neighbourhood of the point $x = a$ has points where $f'(x)$ doesn't exist then $f'(a)$ doesn't exist.

<u>Counter-example</u>.

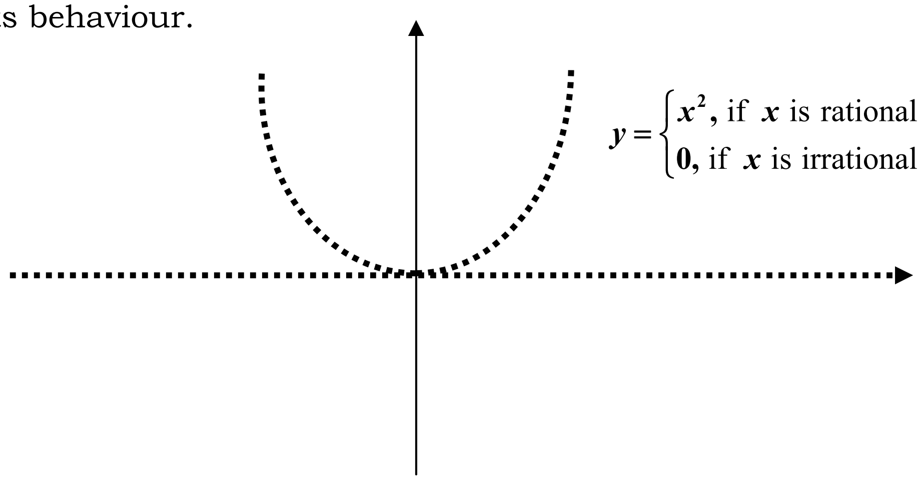
The function $f(x)=\begin{cases}x^2\left|\cos\dfrac{\pi}{x}\right|, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ in any neighbourhood of the point $x = 0$ has points where $f'(x)$ doesn't exist, however $f'(0)=0$.



33. A function cannot be differentiable only at one point in its domain and non-differentiable everywhere else in its domain.

Counter-example.

The function $y = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is defined for all real x and differentiable only at the point $x = 0$. It is impossible to draw the graph of the function $y = f(x)$ but the sketch below gives an idea of its behaviour.

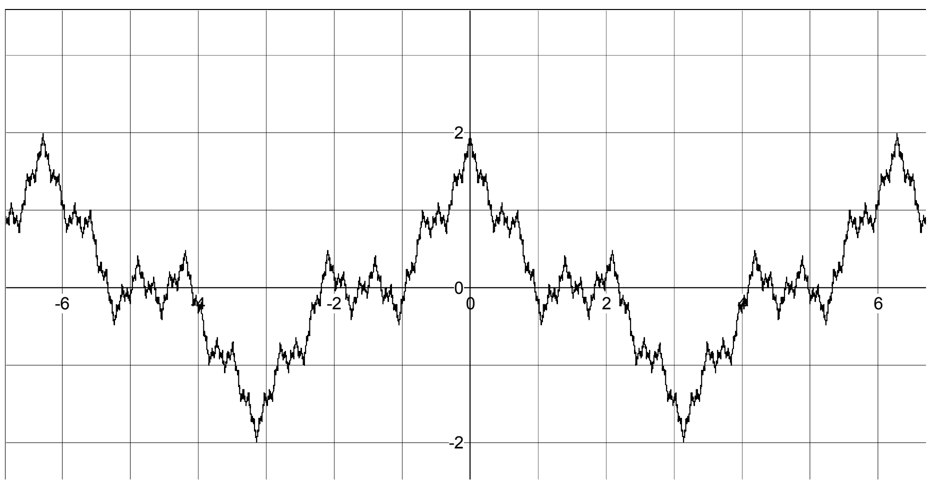


34. A continuous function cannot be non-differentiable at every point in its domain.

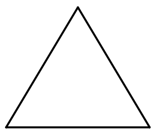
Counter-example.

The Weierstrass' function can be defined as: $f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cos(3^n x)$.

If we take the first 7 terms in the sum we can begin to visualise the function:



Comments. The Weierstrass' function is the first known fractal. Another good example of a continuous curve that has a sharp corner at every point is the Koch's snowflake. We start with an equilateral triangle and build the line segments on each side according to a simple rule and repeat this process infinitely many times. The resulting curve is called Koch's curve and it forms the so-called Koch's snowflake. The first four iterations are shown below:



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5. Integral Calculus

1. If the function $y = f(x)$ is an antiderivative of a function $y = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$.

   Counter-example.

   The function $F(x) = \ln|x|$ is an antiderivative of the function $f(x) = \frac{1}{x}$ but the (improper) integral $\int_{-1}^{1} \frac{1}{x}dx$ doesn't exist.

   Comments. To make the statement true we need to add that the function $y = f(x)$ must be continuous on $[a,b]$.

2. If a function $y = f(x)$ is continuous on $[a,b]$ then the area enclosed by the graph of $y = f(x)$, OX, $x = a$ and $x = b$ numerically equals $\int_a^b f(x)dx$.

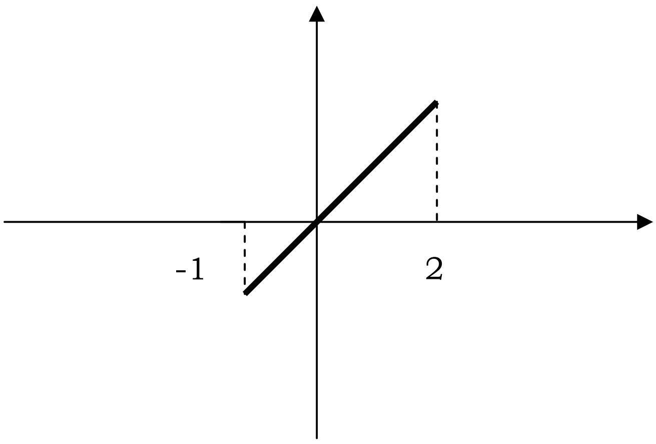
   Counter-example.

   For any continuous function $y = f(x)$ that takes only negative values on $[a,b]$ the integral $\int_a^b f(x)dx$ is negative, therefore the area enclosed by the graph of $f(x)$, OX, $x = a$ and $x = b$ is numerically equal to $-\int_a^b f(x)dx$, or $\left|\int_a^b f(x)dx\right|$.

3. If $\int_a^b f(x)dx \geq 0$ then $f(x) \geq 0$ for all $x \in [a,b]$.

   Counter-example.

   $\int_{-1}^{2} xdx = \frac{3}{2} > 0$ but the function $y = x$ takes both positive and negative values on [-1,2].



4. If $y = f(x)$ is a continuous function and k is any constant then:

$$\int kf(x)dx = k\int f(x)dx \ .$$

<u>Counter-example</u>.

If k = 0 then the left-hand side is: $\int 0f(x)dx = \int 0dx = C$, where C is an arbitrary constant. The right-hand side is: $0\int f(x)dx = 0$. This indicates that C is always equal zero, but this contradicts the nature of an arbitrary constant.

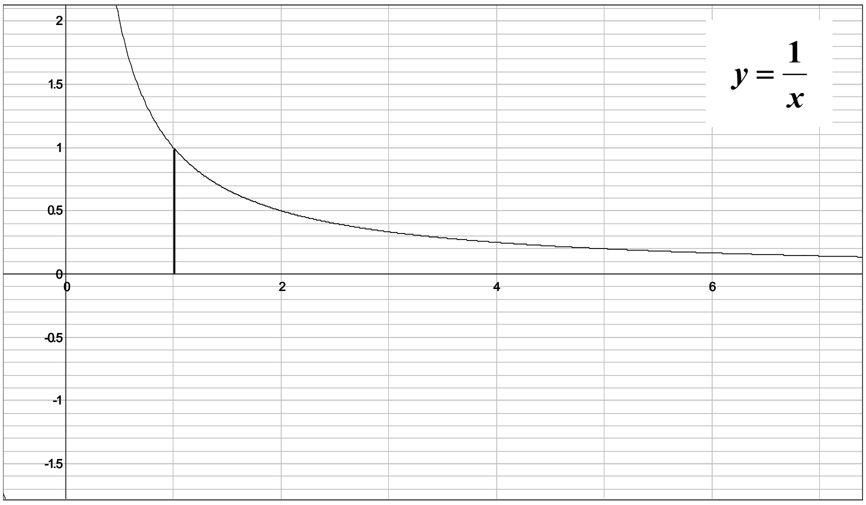
<u>Comments</u>. The property is valid only for non-zero values of the constant k.

5. A plane figure of infinite area rotated around an axis always produces a solid of revolution of infinite volume.

<u>Counter-example</u>.

The figure enclosed by the graph of the function $y = \frac{1}{x}$, the x-axis and the straight line x = 1 is rotated about the x-axis.

The area is infinite: $\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}(\ln b - \ln 1) = \infty$ (square units), but the volume is finite: $\pi\int_1^\infty \frac{1}{x^2}dx = -\pi\lim_{b\to\infty}(\frac{1}{b} - 1) = \pi$ (cubic units).



6. If a function $y = f(x)$ is defined for any $x \in [a,b]$ and $\int_a^b |f(x)|dx$ exists then $\int_a^b f(x)dx$ exists.

<u>Counter-example</u>.

The function $f(x) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ -1, \text{ if } x \text{ is irrational} \end{cases}$ is defined for any real x.

$|f(x)| = 1$ and therefore $\int_a^b |f(x)|dx = b - a$ but $\int_a^b f(x)dx$ does not exist. Let us show this using the definition of the definite integral.

Let $[a,b]$ be any closed interval. We divide the interval into n subintervals and find the limit of the integral sums:

$$S = \lim_{\max \Delta x_i \to 0} \sum_{i=0}^{n-1} f(c_i)\Delta x_i \ .$$

If on any subinterval we choose c_i equal to a rational number then $S = b - a$. If on any subinterval we choose c_i equal to an irrational number then $S = a - b$. So the limit of the integral sums depends on the way we choose c_i and for this reason the definite integral of $f(x)$ on $[a,b]$ doesn't exist.

7. If neither of the integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist then the integral $\int_a^b (f(x)+g(x))dx$ doesn't exist.

<u>Counter-example</u>.

For the functions

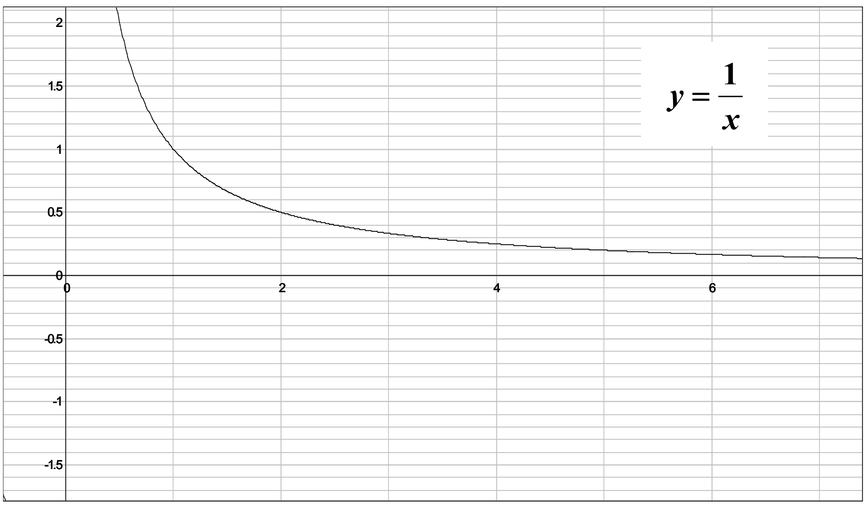
$$f(x)=\begin{cases} 1, \text{ if } x \text{ is rational} \\ -1, \text{ if } x \text{ is irrational} \end{cases} \quad \text{and} \quad g(x)=\begin{cases} -1, \text{ if } x \text{ is rational} \\ 1, \text{ if } x \text{ is irrational} \end{cases}$$

the integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ don't exist (see the previous exercise) but the integral $\int_a^b (f(x)+g(x))dx$ exists and equals 0.

8. If $\lim_{x\to\infty} f(x)=0$ then $\int_a^\infty f(x)dx$ converges.

<u>Counter-example</u>.

The limit $\lim_{x\to\infty}\frac{1}{x}=0$ but the integral $\int_1^\infty \frac{1}{x}dx$ diverges.



9. If the integral $\int_a^{\infty} f(x)dx$ diverges then the function $y = f(x)$ is not bounded.

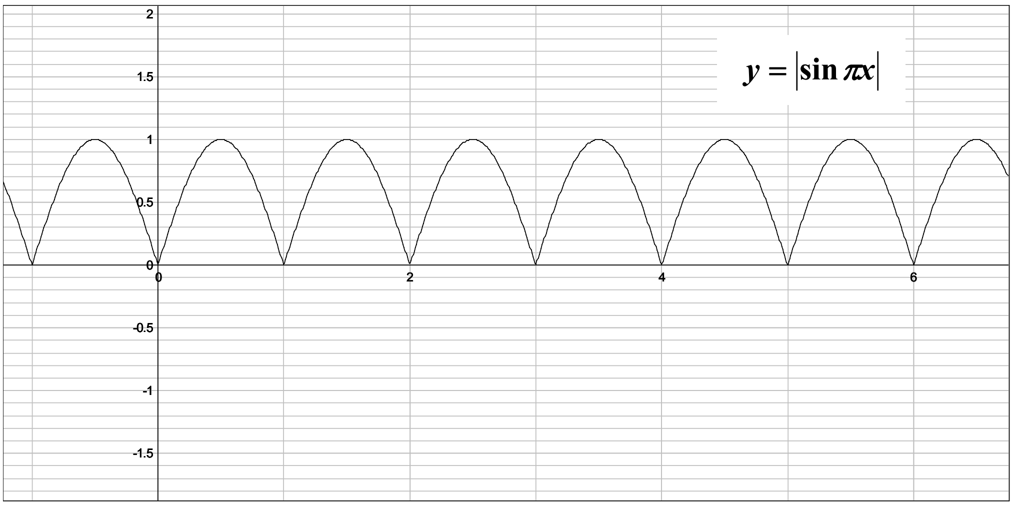
Counter-example.

The integral of a non-zero constant $\int_a^{\infty} k\,dx$ is divergent but the function $y = k$ is bounded.

10. If a function $y = f(x)$ is continuous and non-negative for all real x and $\sum_{n=1}^{\infty} f(n)$ is finite then $\int_1^{\infty} f(x)dx$ converges.

Counter-example.

The function $y = |\sin \pi x|$ is continuous and non-negative for all real x and $\sum_{n=1}^{\infty} |\sin \pi n| = 0$ but $\int_1^{\infty} |\sin \pi x| dx$ diverges.



11. If both integrals $\int_a^{\infty} f(x)dx$ and $\int_a^{\infty} g(x)dx$ diverge then the integral $\int_a^{\infty} (f(x)+g(x))dx$ also diverges.

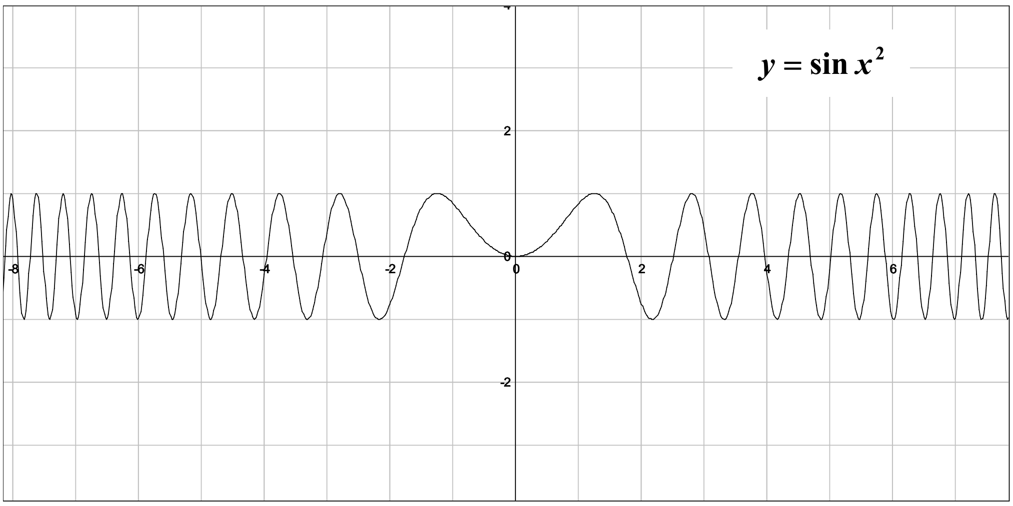
Counter-example.

Both integrals $\int\limits_1^\infty \frac{1}{x}dx$ and $\int\limits_1^\infty \frac{1-x}{x^2}dx$ diverge but the integral $\int\limits_1^\infty (\frac{1}{x}+\frac{1-x}{x^2})dx = \int\limits_1^\infty \frac{1}{x^2}dx$ converges.

12. If a function $y = f(x)$ is continuous and $\int\limits_a^\infty f(x)dx$ converges then $\lim\limits_{x\to\infty} f(x) = 0$.

Counter-example.

The Fresnel integral $\int\limits_0^\infty \sin x^2 dx$ converges but $\lim\limits_{x\to\infty} \sin x^2$ does not exist:

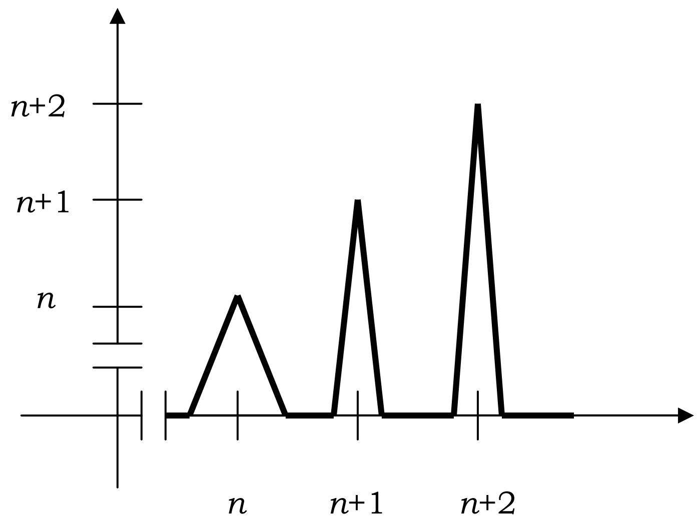


13. If a function $y = f(x)$ is continuous and non-negative and $\int\limits_a^\infty f(x)dx$ converges then $\lim\limits_{x\to\infty} f(x) = 0$.

Counter-example.

We will use the idea of area. Over every natural n we can construct triangles of area $\frac{1}{n^2}$ so that the total area equals $\sum\limits_{n=a}^{\infty} \frac{1}{n^2}$, which is a finite number. The height of each triangle is n and the base is $\frac{2}{n^3}$.

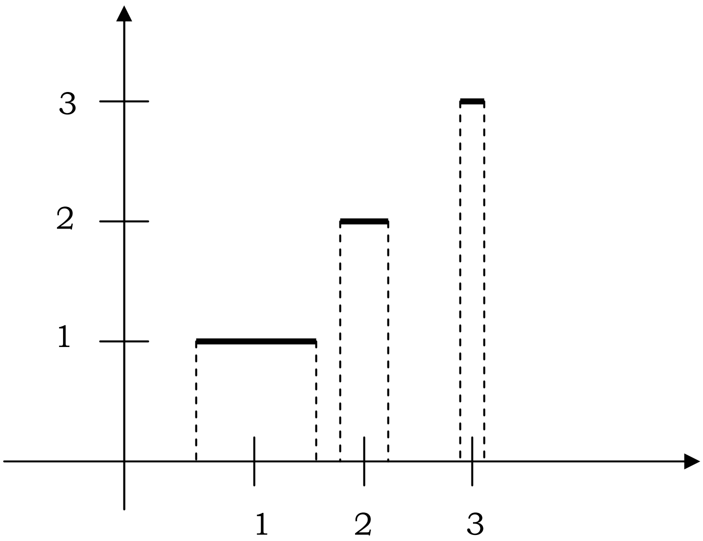
The integral $\int_a^{\infty} f(x)dx$ converges since it is numerically equal to the total area $\sum_{n=a}^{\infty} \frac{1}{n^2}$. As one can see from the graph below the function (in bold) is continuous and non-negative but $\lim_{x\to\infty} f(x)$ doesn't exist.



14. If a function $y = f(x)$ is positive and unbounded for all real x then the integral $\int_a^{\infty} f(x)dx$ diverges.

Counter-example.

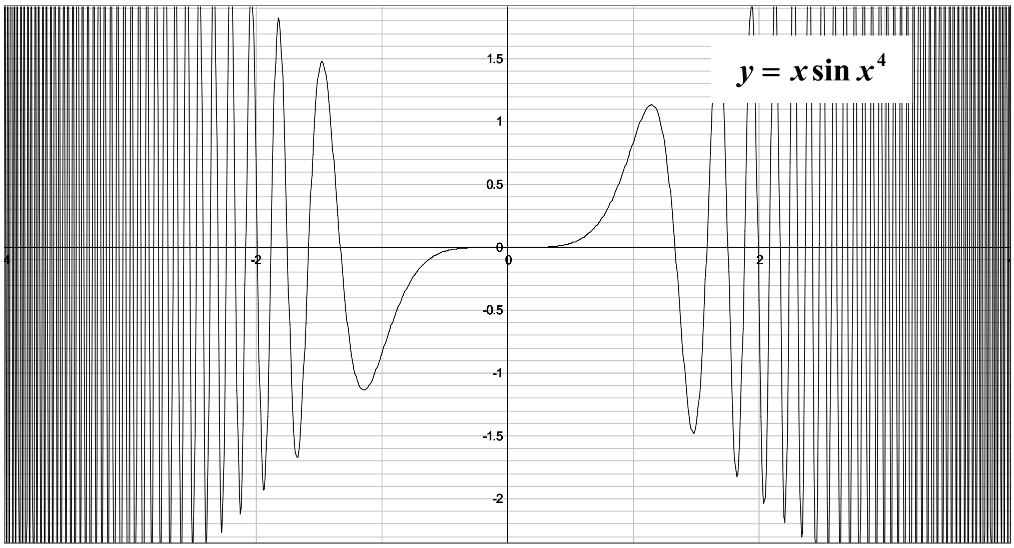
We will use the idea of area. Over every natural n we can construct a rectangle with the height n and the base $\frac{1}{n^3}$ so the area is $\frac{1}{n^2}$. Then the total area equals $\sum_{n=a}^{\infty} \frac{1}{n^2}$, which is a finite number. The positive and non-bounded function equals n on the interval of length $\frac{1}{n^3}$ around points $x = n$, where n is natural. Since the integral $\int_a^{\infty} f(x)dx$ numerically equals the total area $\sum_{n=a}^{\infty} \frac{1}{n^2}$ it converges.



15. If a function $y = f(x)$ is continuous and not bounded for all real x then the integral $\int_0^\infty f(x)dx$ diverges.

<u>Counter-example</u>.

The function $y = x\sin x^4$ is continuous and unbounded for all real x, but the integral $\int_0^\infty x\sin x^4 dx$ converges (making the substitution $t = x^2$ yields the Fresnel integral $\frac{1}{2}\int_0^\infty \sin t^2 dt$ which is convergent).



16. If a function $y = f(x)$ is continuous on $[1,\infty)$ and $\int_1^\infty f(x)dx$ converges then $\int_1^\infty |f(x)|dx$ also converges.

Counter-example.

The function $y = \frac{\sin x}{x}$ is continuous on $[1,\infty)$ and $\int_1^\infty \frac{\sin x}{x}dx$ converges but $\int_1^\infty \left|\frac{\sin x}{x}\right|dx$ diverges.

17. If the integral $\int_a^\infty f(x)dx$ converges and a function $y = g(x)$ is bounded then the integral $\int_a^\infty f(x)g(x)dx$ converges.

Counter-example.

The integral $\int_0^\infty \frac{\sin x}{x}dx$ converges and the function $g(x) = \sin x$ is bounded but the integral $\int_0^\infty \frac{\sin^2 x}{x}dx$ diverges.

Comments. Statements 10, 13 and 14 in this chapter are supplied by Alejandro S.Gonzalez-Martin, University La Laguna, Spain.

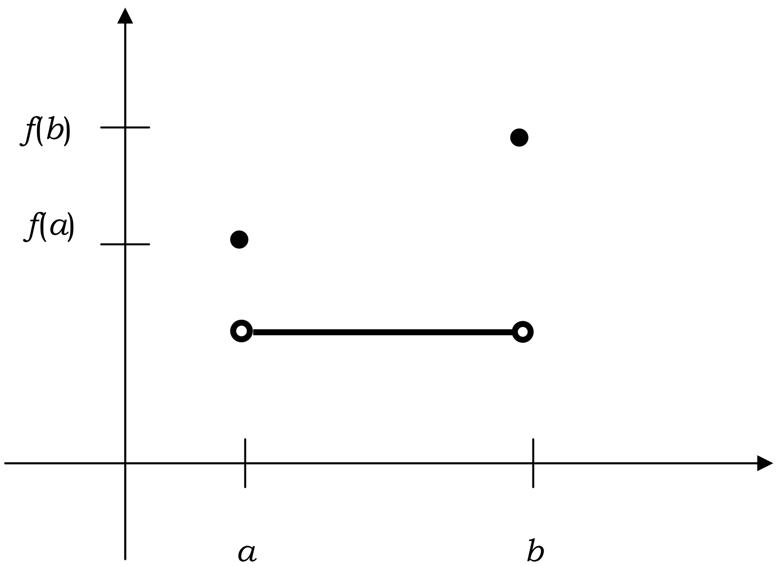
Appendix 1.

Example from Teaching Practice

Below are some thoughts and suggestions from my experience using counter-examples with students. Statement 14 from Continuity is considered as an example:

<u>Statement.</u> If a function $y = f(x)$ is defined on $[a,b]$ and continuous on (a,b) then for any $N \in (f(a), f(b))$ there is some point $c \in (a,b)$ such that $f(c) = N$.

The only difference between this statement and the Intermediate Value Theorem is that continuity of the function is required on an open interval (a,b), instead of a closed interval $[a,b]$. In other words, one side continuity of the function at the point $x = a$ from the right and at the point $x = b$ from the left is not required. When students are asked to disprove the statement they usually come up with something like this:



To generate discussion and create other counter-examples one can suggest that:

In the above graph the statement's conclusion is not true for any value of $N \in (f(a), f(b))$. *Modify the graph in such a way that the statement's conclusion is true for:*

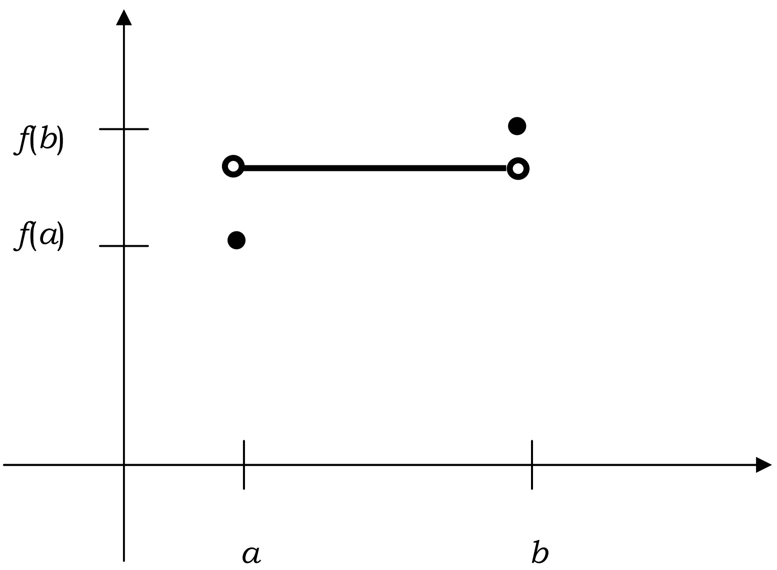
a) *one value of* $N \in (f(a), f(b))$

b) *two values of* $N \in (f(a), f(b))$

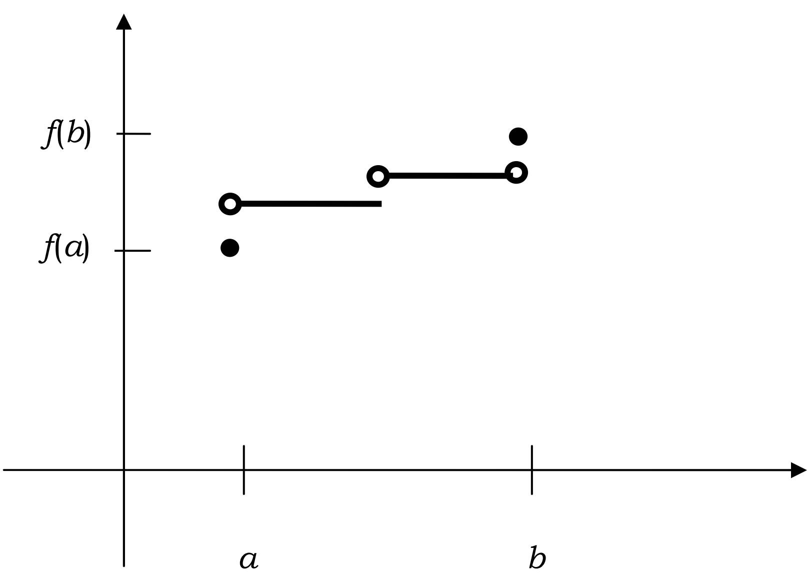
c) *infinitely many but not all values of* $N \in (f(a), f(b))$.

One can expect from the students the following three sketches:

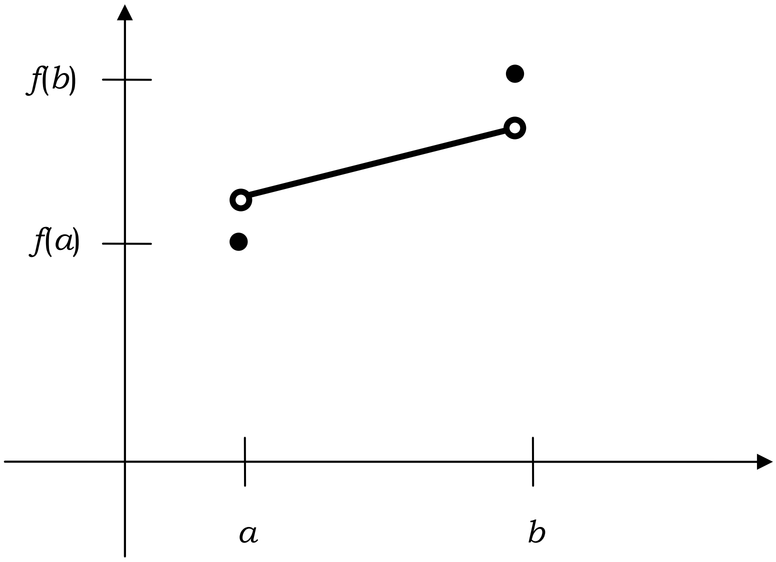
a)



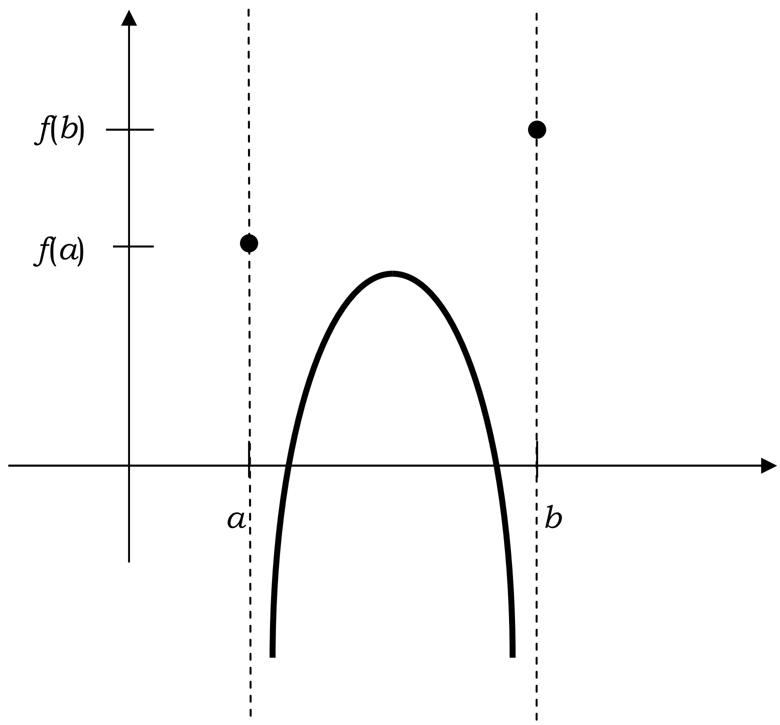
b)



c)



Another challenge can be presented:
Give a counter-example such that the graph doesn't have white circles.
In this case students may come up with something like this:



Some students find such problems very new and challenging. After learning Calculus at school many come to university with a strong preference to performing calculations, manipulations and techniques, ignoring conditions of the theorems and properties of the functions they are dealing with. It is often not their fault.

To illustrate the point, below is a real example from a final year high-school mathematics exam (university entrance) that deals with the Intermediate Value Theorem as well.

Question. "Show that the equation $x^2 - \sqrt{x} - 1 = 0$ has a solution between $x = 1$ and $x = 2$.
Model Solution. If $f(x) = x^2 - \sqrt{x} - 1$ then $f(1) = -1 < 0$ and $f(2) = 1.58 > 0$. So graph of f crosses the x-axis between 1 and 2."

The above model solution was given to examiners as a complete solution, one for which students would get full marks for. It was based on the special case of the Intermediate Value Theorem which has 2 conditions: the continuity of $f(x)$ on $[a,b]$ and the condition $f(a) \times f(b) < 0$. Only the second condition was checked, and the first was ignored as if it was 'not essential'. The question came from a written exam where all working had to be shown. The fact that the condition of continuity of the function $f(x)$ was not required by the examiners to award full marks for the solution was very dangerous. The message was clear – calculations are important but the function's properties are not. No wonder students don't consider all theorem conditions and properties of functions – it is simply not required.

Some years ago I gave the following provocative question to five average students who had done some Calculus at school and at that time were taking a University entrance mathematics course: "Show that the equation $\dfrac{x^2 + x + 1}{x - 1.5} = 0$ has a solution between $x = 1$ and $x = 2$." All five students quickly 'showed' this by misusing the Intermediate Value Theorem. They only checked that $f(1) = -6 < 0$ and $f(2) = 14 > 0$. This particular function was chosen deliberately over the simple hyperbola $f(x) = \dfrac{1}{x - 1.5}$ to provoke the students to jump straight into calculations, which they did. It's hard to blame them for this if they are mainly used to performing calculations, manipulations and techniques in school mathematics.

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Using Counter-Examples in Teaching Calculus: Students' Attitudes

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ABSTRACT: The paper deals with a practical issue encountered by many lecturers teaching first-year university Calculus. A big proportion of students seem to be able to find correct solutions to test and exam questions using familiar steps and procedures. Yet they lack deep conceptual understanding of the underlying theorems and sometimes have misconceptions. In order to reduce or eliminate misconceptions, and for deeper understanding of the concepts involved, the students were given the incorrect mathematical statements and were asked to construct counter-examples to disprove the statements. More than 600 students from 10 universities in different countries were questioned regarding their attitudes towards the method of using counter-examples for eliminating misconceptions and deeper conceptual understanding. The vast majority of the students reported that the method was very effective and made learning mathematics more challenging, interesting and creative.

INTRODUCTION AND FRAMEWORK

In the information age analysing given information and making a quick decision on whether it is true or false is an important ability. A counter-example is an example that shows that a given statement (conjecture, hypothesis, proposition, rule) is false. It only takes one counter-example to disprove a statement. Counter-examples play an important role in mathematics and other subjects. They are a powerful and effective tool for scientists, researchers and practitioners. They are good indicators showing that the suggested hypothesis or chosen direction of research is wrong. Before trying to prove the conjecture or hypothesis it is often worth looking for a possible counter-example. This can save lots of time and effort. We decided to introduce this powerful method to our students. Creating examples and counter-examples is neither algorithmic nor procedural and requires advanced mathematical thinking, things not often taught in schools [15, 19, 20]. As Seldens write, "coming up with examples requires different cognitive skills from carrying out algorithms – one needs to look at mathematical objects in terms of their properties. To be asked for an example can be disconcerting. Students have no pre-learned algorithms to show the 'correct way' " [15]. Many students are used to concentrating on techniques, manipulations and familiar procedures without paying much attention to the concepts, conditions of the theorems, properties of the functions, nor to the reasoning and justification behind them.

Over recent years in some countries, partly due to extensive usage of modern technology, the proof component of the traditional approach in teaching mathematics to engineering students (definition-theorem-proof-example-application) has almost disappeared. Students are used to relying on technology and sometimes lack logical thinking and conceptual understanding. Sometimes mathematics courses, especially at school level, are taught in such a way that special cases are avoided and students are exposed only to 'nice' functions and 'good' examples. This approach can create many misconceptions, explained by Tall's generic extension principle: "If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts." [18]. "The rapid increase of information over very short periods of time is a major problem in engineering education that seems worldwide. Misconceptions or unsuitable preconceptions cause many difficulties" [2]. "The basic knowledge, performance and conceptual understanding of the students in mathematics worsen" [3].

The study's main objective was to verify our assumptions regarding the effectiveness of counter-examples in giving students deeper conceptual understanding, eliminating their misconceptions and developing a creative learning environment for teaching/learning university Calculus.

In this study, practice was selected as the basis for the research framework, and it was decided "to follow conventional wisdom as understood by the people who are stakeholders in the practice" [1]. The theoretical framework was based on Piaget's notion of cognitive conflict [4]. Some studies in mathematics education at school level [5], [6] found conflict to be more effective than direct instruction. "Provoking cognitive conflict to help students understand areas of mathematics is often recommended" [6]. Swedosh and Clark [7] used conflict in their intervention method to help undergraduate students eliminate their misconceptions. "The method essentially involved showing examples for which the misconception could be seen to lead to a ridiculous conclusion, and, having established a conflict in the minds of the students, the correct concept was taught" [7]. Another study by Horiguchi and Hirashima [14] used a similar approach in creating a 'discovery learning environment' in their mechanics classes. They showed counter-examples to their students and considered them a chance to learn from their mistakes. They claim that for counter-examples to be effective they "must be recognized to be meaningful and acceptable and must be suggestive, to lead a learner to correct understanding" [14]. Mason and Watson [8] used a method of so-called 'boundary examples', which suggested students come up with examples to correct statements, theorems, techniques, and questions that satisfied their conditions. "When students come to apply a theorem or technique, they often fail to check that the conditions for applying it are satisfied. We conjecture that this is usually because they simply do not think of it, and this is because they are not fluent in using appropriate terms, notations, properties, or do not recognise the role of such conditions" [8]. In our study, the students, not the lecturers, were asked to create and show counter-examples to the incorrect statements, i.e. the students themselves established a conflict in their minds. The students were actively involved in creative discovery learning that stimulated development of their advanced mathematical thinking.

THE STUDY

To develop a creative learning environment, enhance students' critical thinking skills, help them understand concepts and theorems' conditions better, reduce or even eliminate common misconceptions and encourage active participation in class, the students were given incorrect statements and asked to create counter-examples to prove that the statements were wrong. They had enough knowledge to do this, but for most of the students this kind of activity was entirely new and challenging, and even created psychological discomfort and conflict for a number of reasons. In the beginning some of the students could not see the difference between 'proving' that the statement is correct by an example and disproving it by an example. It agrees with the following quotation from Seldens [15]: "Students quite often fail to see a single counter-example as disproving a conjecture. This can happen when a counter-example is perceived as 'the only one that exists', rather than being seen as generic".

In our study we did not use 'pathological' cases. All the wrong statements given to the students were within their knowledge and often related to their common misconceptions.

Below are 6 examples of such statements that were discussed with the students:

- With a continuous function, i.e. a function which has values of y which smoothly and continuously change for all values of x, we have derivatives for all values of x.
- If the first derivative of a function is zero at a point then the function is neither increasing nor decreasing at this point.
- At a maximum the second derivative of a function is negative and at a minimum positive.

- The tangent to a curve at a point is the line which touches the curve at that point but does not cross it there.
- If $F(x)$ is defined on $[a,b]$ and continuous on (a,b) then for any N between $F(a)$ and $F(b)$ there is some point c between a and b for which $F(c) = N$.
- If the absolute value of $F(x)$ is continuous on (a,b) then $F(x)$ is continuous on (a,b).

The first four out of the above 6 statements are quotations from university calculus textbooks published by reputable publishers. In this paper we will not look into the issue of mistakes in textbooks and their effect on students learning mathematics, as it is a sensitive topic. We can refer readers to a case study done by one of the co-authors [11]. We mention this here due to the importance of developing and enhancing critical thinking by students for analyzing any information - not only printed in newspapers but also in mathematics textbooks. In addition, some such textbooks might be used as a good resource for students for finding incorrect statements on the given pages and creating counter-examples to them.

After several weeks of using counter-examples in teaching Calculus to first-year engineering students, 612 students from 10 universities in different countries were given the questionnaire below to investigate their attitudes towards the usage of counter-examples in learning/teaching. A combination of two non-probability sampling methods - judgement and convenience – was used to select lecturers who conducted a survey with students in their universities. The survey was sent to selected participants of international mathematics education conferences held in 2000-2002 and university lecturers who either teach university mathematics or write papers on mathematics education at university level, or both. The response rate was 60% which demonstrates the importance of the topic - this is quite a high rate of response for busy professionals. A cross-cultural approach was chosen to reduce the effect of differences in education systems, curricula, cultures and also to analyse the data from different perspectives and backgrounds. The questionnaire is below.

THE QUESTIONNAIRE

Question 1. Do you feel confident using counter-examples?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 2. Do you find this method effective?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 3. Would you like this kind of activity to be a part of assessment?

a) Yes Please give the reasons:

b) No Please give the reasons:

FINDINGS FROM THE QUESTIONNAIRE

The statistics from the questionnaire are presented in the following table:

Number of Students	Question 1 Confident?		Question 2 Effective?		Question 3 Assessment?	
	Yes	No	Yes	No	Yes	No
612	116	496	563	49	196	416
100%	19%	81%	92%	8%	32%	68%

Table 1. Summary of findings from the questionnaire

The majority of the students (81%) were not familiar with the usage of counter-examples as a method of disproof. The typical comments from students who answered "No" to question 1 (confidence) were as follows:

- I have never done this before;
- I am not familiar with this at all;
- I am not used to this method;
- this method is unknown to me;

- we did not learn it at school;
- I heard about it but not from my school teacher;
- we hardly created ourselves any examples at school.

The vast majority of the students (92%) found the method of using counter-examples to be effective. The typical comments from the students who answered "Yes" to question 2 (effectiveness) were as follows:

- helps me to think about questions deeply;
- gives more sound knowledge of the subject;
- we can understand more;
- it makes me think more effectively;
- can prevent mistakes;
- you gain a better understanding;
- it makes the problem more clear;
- it boosts self-confidence;
- it helps you retain information that you have learned;
- it is a good teaching tool;
- it teaches you to question everything;
- it makes you think carefully about the concepts and how they are applied;
- it makes you think critically;
- it supports self-control;
- it requires logical thinking, not only calculations;
- makes problems more understandable;
- it is hard but it is fun;
- it is a good way to select top students;
- I can look at maths from another angle;
- it is good not only in mathematics;
- it really forces you to think hard;
- it is not a routine exercise, it is creative.

The majority of the students (68%) did not want the questions on creating counter-examples to incorrect statements to be part of assessment, in contrast to the trends pointing to the effectiveness of the method (92%).

The typical comments from the students who answered "No" to question 3 (assessment) were generally as follows:

- it is hard;
- never done this stuff before;
- confusing;
- not trained enough;
- complicated;
- not structured;
- not enough time to master it;
- you don't know how to start;
- can affect marks.

The last comment was the most frequent. The majority of these students were more concerned about their test results than acquiring useful skills. Apparently their attitudes towards learning still had some maturing to do.

The students who answered "Yes" (32%) provided excellent comments similar to those made on the method's effectiveness. The comments from the students who answered "Yes" to question 3 (assessment) were as follows:

- it provokes generalised thinking about the nature of the processes involved, as compared to the detail of the processes;
- better performance test;
- it shows full understanding of topic;
- a good way to test students' insight;
- it is an extremely valuable skill;
- it is good to have it in assessment otherwise we will not put much attention to it;
- it is challenging and I like it;
- one can use this method outside university;

CONCLUSIONS AND RECOMMENDATIONS

The statistical results of the study and the numerous comments from students confirm that the students were positive about the usage of counter-examples in first-year undergraduate mathematics. Many of them reported that the use of counter-examples helped them to understand concepts better, prevent mistakes in working, develop logical and critical thinking, and make their participation in lectures more active. All these give us confidence to recommend this pedagogical strategy to our colleagues to try with their students. There are different ways of using this strategy: giving the students a mixture of correct and incorrect statements; making a deliberate mistake in the lecture; asking the students to spot an error on a certain page of their textbook or manual; giving the students extra (bonus) marks towards their final grade for providing excellent counter-examples to hard questions during the lecture and so on. At more advanced levels of mathematics Dahlberg and Housman suggest "it might be beneficial to introduce students to new concepts by having them generate their own examples or having them decide whether teacher-provided candidates are examples or non-examples, before providing students examples and explanations" [13].

Many students commented that creating counter-examples was closely connected with enhancing their critical thinking skills. These skills are general ones and can be used by the students in other areas of their life that have nothing to do with mathematics. The ability to create counter-examples is an important instrument of critical selection in the broader sense. Henry Perkinson, the author of the famous book "Learning from our mistakes" [16], writes about the importance of those skills for his theory of education in his recent publication: "Our knowledge is imperfect, it can always get better, improve, grow. Criticism facilitates this growth. Criticism can uncover some of the inadequacies in our knowledge, and when we eliminate them, our knowledge evolves and gets better…Education is a continual process of trial-and-error elimination. Students are fallible creators who make trial conjectures and formulate trial skills and then eliminate the errors uncovered by criticism and critical selection" [17].

FURTHER STUDY

We would like to extend the study to measure the effectiveness of this pedagogical strategy on the students' exam performance on the questions that require good understanding of concepts, not just skills related to manipulation and technique. We plan to compare the performance of 2 groups of students with similar backgrounds. In one group we will use counter-examples extensively, with the other group being the control group. Then we will use statistical methods to establish whether the difference is significant or not.

We also plan to develop a database of incorrect statements related to common students' misconceptions for practice in creating counter-examples. This supplementary teaching resource could be used for both teaching and assessment.

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Counter-Examples in Teaching/Learning of Calculus: Students' Performance

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ABSTRACT: This paper presents a case study that involved first year science and engineering students at the Auckland University of Technology, New Zealand. After very positive feedback was received from students in the international study on their attitudes towards counter-examples in the teaching/learning of Calculus (Gruenwald, Klymchuk, 2003) it was decided to investigate how the use of counter-examples would affect students' performance. The case study showed that the usage of counter-examples significantly improved students' performance on a test question that required conceptual understanding, but did not affect their performance on other test questions that only required the application of familiar rules, algorithms and calculations.

INTRODUCTION

Counter-examples provide an important means of communicating ideas in mathematics, whose entire history may be viewed as making conjectures and then either proving or disproving them by counter-example. Here are a few well-known cases to illustrate the point:

1. For a long time mathematicians tried to find a formula for prime numbers. The numbers of the form $2^{2^n}+1$, where n is natural were once considered as prime numbers, until a counter-example was found. For n = 5 that number is composite: $2^{2^5}+1=641\times 6700417$.

2. Another conjecture about prime numbers is still waiting to be proved or disproved - Goldbach's or the Goldbach-Euler conjecture, posed by Goldbach in his letter to Euler in 1742. It looks deceptively simple at first. It states that *every even number greater than 2 is the sum of 2 prime numbers*. For example, 12 = 5 + 7, 20 = 3 + 17, and so on. A powerful computer was used in 1999 to search for counter-examples to that conjecture. No counter-examples have been found up to 4×10^{14}. In 2000 the book publishing company Faber & Faber offered a US$1 million prize to anyone who could prove or disprove that conjecture. To date (April, 2005) the prize remains unclaimed.

3. In the 19^{th} century the great German mathematician Weierstrass constructed his famous counter-example – the first known fractal – to the statement: *a function continuous on* (a,b) *cannot be non-differentiable at any point on* (a,b). Many mathematicians at that time thought that such 'monster-functions' that were continuous but not differentiable at any point were absolutely useless for practical applications. About a hundred years later Norbert Winer, the founder of cybernetics pointed out in his book "I am a mathematician" that such curves exist in nature – for example, they are trajectories of particles in Brownian motion. In recent decades such curves have been investigated in the theory of fractals – a fast growing area with many applications.

Using counter-examples in teaching/learning of Calculus can be beneficial in many areas:

- For deeper conceptual understanding
- To reduce or eliminate common misconceptions
- To advance one's mathematical thinking, that is neither algorithmic nor procedural

- To enhance generic critical thinking skills – analysing, justifying, verifying, checking, proving which can benefit students in other areas of life
- To expand the 'example set' - a number of examples of interesting functions for better communication of ideas in mathematics and in practical applications
- To make learning more active and creative

The international study on students' attitudes towards the usage of counter-examples as a pedagogical strategy in the teaching/learning of Calculus (Gruenwald, Klymchuk, 2003) involving over 600 students from 10 universities around the world showed students' attitudes to be very positive. 92% of the participants reported that this strategy was very effective. Many of them commented that it helped them to understand concepts better, prevent mistakes, develop logical and critical thinking skills, and that it made their participation in lectures more active. Students' attitudes and their exam performance are different matters though, so this study investigates how the use of counter-examples affects students performance.

THE STUDY

Two groups of students enrolled in science or engineering courses from the Auckland University of Technology were selected for this case study. In group A there were 14 students and in group B (the control group) there were 11. All of the students had similar mathematics backgrounds and ages, and all were Chinese. Both groups attended 3 lectures and 1 tutorial per week with the same lecturer, except group A was taught by another lecturer once a week who spent 5-6 minutes (out of a 50 minute lecture) on counter-examples. There were 8 weeks before the mid-semester test, so counter-examples were used in a total of 8 lectures. During this 8-week period the entire time spent on counter-examples in the lectures was about 45 minutes.

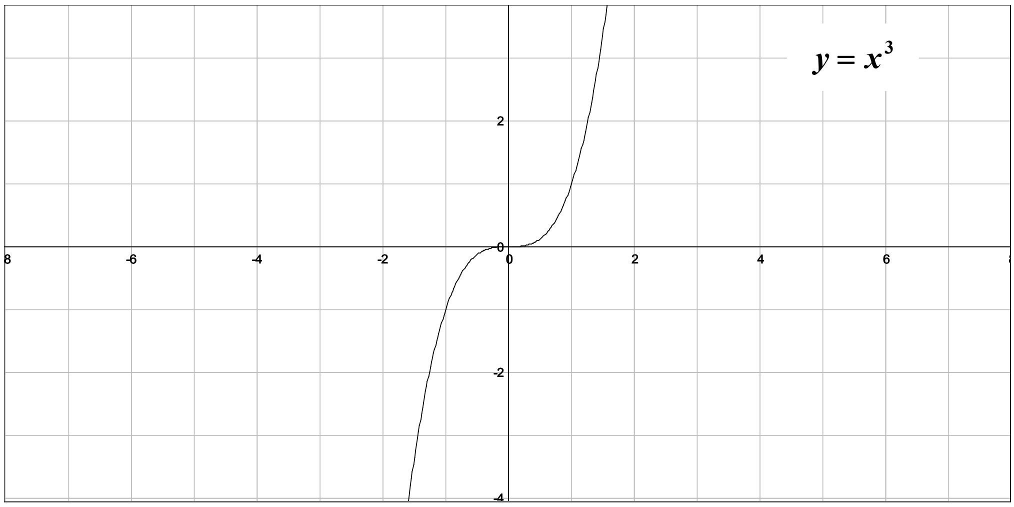
Below are some statements and related counter-examples that were discussed in the group A's lectures.

Statement 1

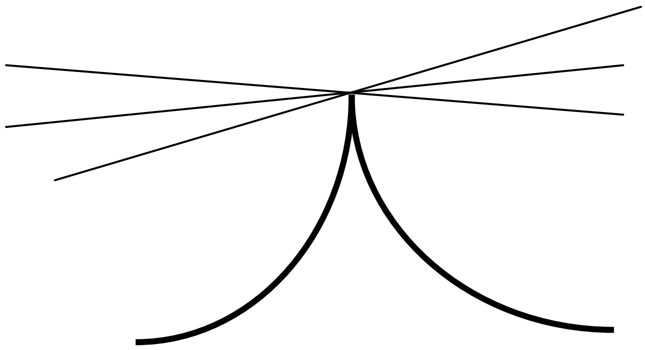
The tangent to a curve at a point is the line which touches the curve at that point but does not cross it there.

Counter-example

a) The x-axis is the tangent line to the curve $y = x^3$ but it crosses the curve at the origin.



b) The three straight lines just touch but don't cross the curve below at the peak, but none of them is the tangent line to the curve at that point.



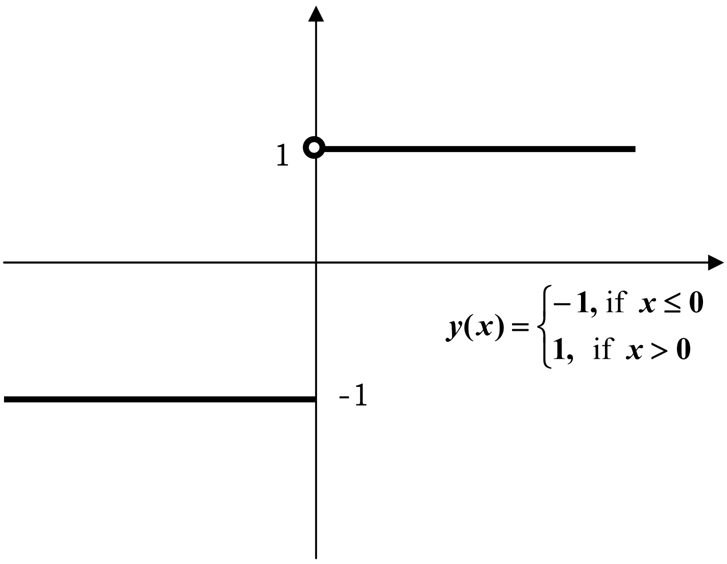
Statement 2

If the absolute value of the function $y = f(x)$ is continuous on (a,b) then the function is also continuous on (a,b).

Counter-example

The absolute value of the function

$y(x)=\begin{cases}-1, \text{ if } x\le 0\\ 1, \text{ if } x>0\end{cases}$ is $|y(x)|=1$ for all real x and it is continuous, but the function $y(x)$ is discontinuous.

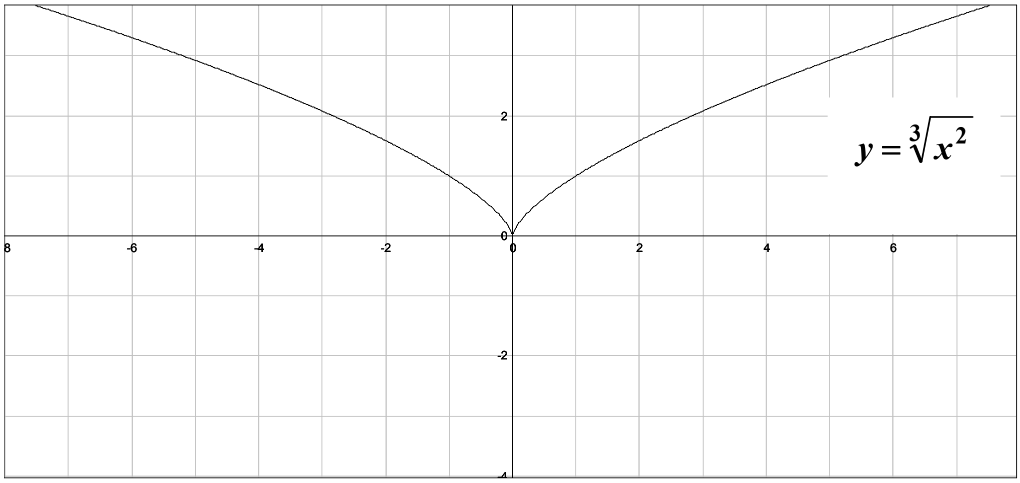


Statement 3

If a function is continuous on R and the tangent line exists at any point on its graph then the function is differentiable at any point on R.

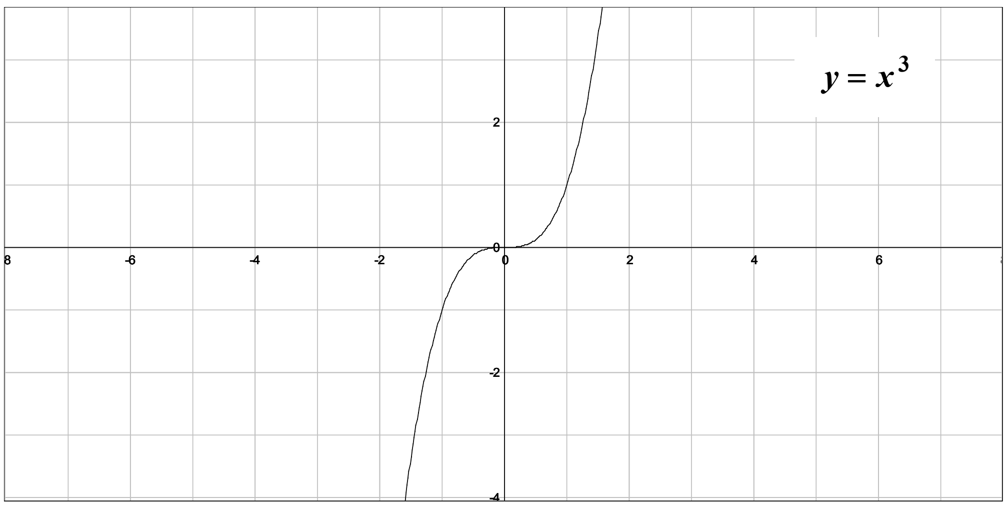
Counter-example

The function $y=\sqrt[3]{x^2}$ is continuous on R and the tangent line exists at any point on its graph but the function is not differentiable at the point x = 0.



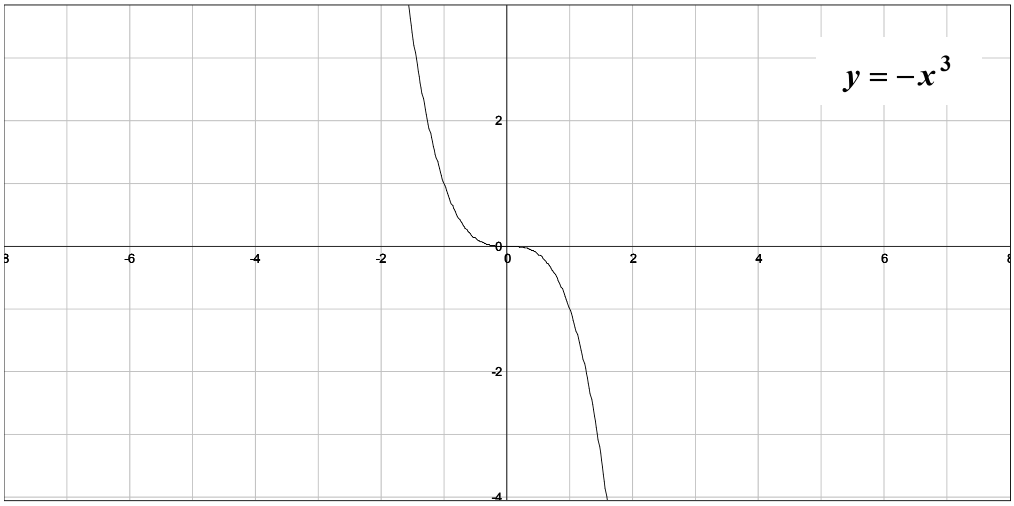
Statement 4
If the derivative of a function is zero at a point then the function is neither increasing nor decreasing at this point.

Counter-example
The derivative of the function $y = x^3$ is zero at the point x = 0 but the function is increasing at this point.



Statement 5
If a function is differentiable and decreasing on (a,b) then its gradient is negative on (a,b).

Counter-example
The function $y = -x^3$ is differentiable and decreasing on R but its gradient is zero at the point x = 0.



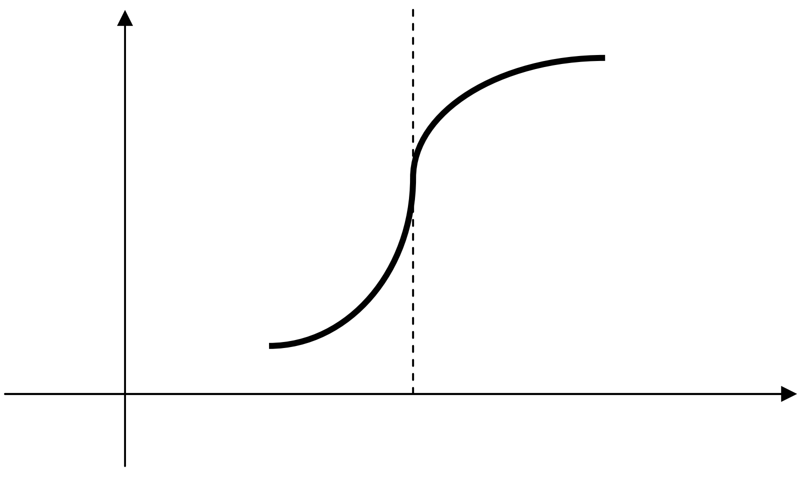
The purpose of this experiment was to see how using counter-examples in class affected students' performance on the test question that required conceptual understanding.

THE RESULTS

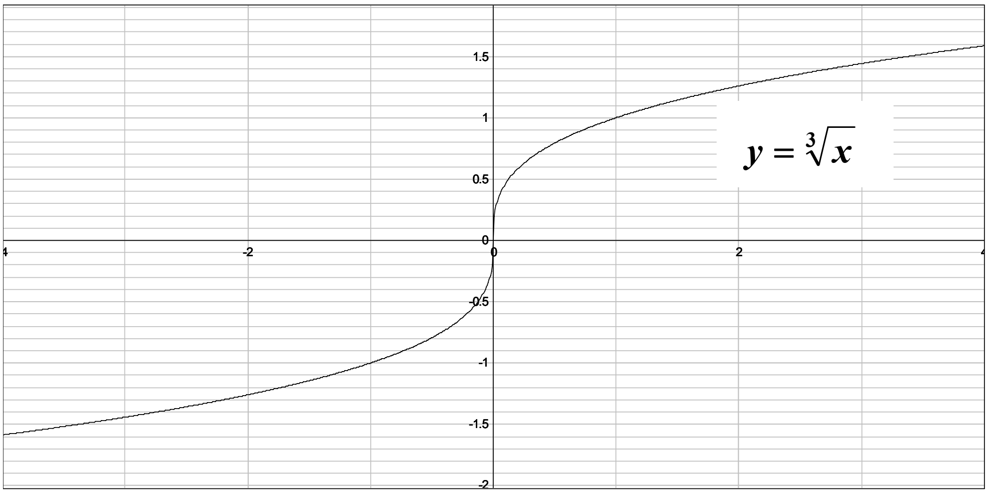
After 8 weeks of study both groups sat the same mid-semester test containing 11 questions: the first 10 questions dealt with skills in techniques, and question 11 tested conceptual understanding.

<u>Question 11</u>. *Sketch a graph of a function that is continuous and smooth* (*no sharp corner*) *at a point but which is not differentiable at that point.*

What we expected from the students was a simple sketch:



Or perhaps something like the cube root function:



The results of the test and question 11 are below:

Group A: Passed the test 13/14=93% Question 11: 11/14=79%
Group B: Passed the test 10/11=91% Question 11: 5/11=45%

DISCUSSION AND CONCLUSION

The students' performance on questions 1-10 was very similar in both groups, as were their overall test results: 93% of the students in group A and 91% in group B passed (received more that 50% of the total marks). When looking at the results of question 11, a significant difference between the two groups is apparent. 79% of the students in group A answered question 11 correctly, versus 45% in group B. This might suggest that group A's conceptual understanding was improved as an immediate result of their work with counter-examples.

As with any case study an essential question is this: to what extent can we generalise these results? Regardless of the answer, employing counter-examples as a pedagogical strategy is certainly worth trying!

There is a well-known book on counter-examples in Calculus: "Counterexamples in Analysis" by B.*R*.Gelbaum and J.M.H.Olmsted (Holden-Day, Inc., San Francisco, 1964). It is an excellent resource for the teaching and learning of Calculus at an advanced level, but it is well beyond the scope of first-year university Calculus courses, ones that might be based on the popular "Calculus: Concepts and Contexts" by

J. Stewart (Brooks/Cole, Thomson Learning, 2nd ed., 2001) for example. Another supplementary teaching resource is the recently published book "Counter-Examples in Calculus" (Klymchuk, 2004). These two books don't overlap – all statements and examples are different. The latter book is aimed at filling the niche in the activity on using counter-examples as a pedagogical strategy in teaching/learning of a first-year university Introductory Calculus course.

REFERENCES

Gruenwald, N. & Klymchuk, S., Using counter-examples in teaching Calculus. *The New Zealand Mathematics Magazine* 40, 2, 33-41 (2003).

Klymchuk, S., *Counter-Examples in Calculus*. Maths Press. Auckland, New Zealand, ISBN 0-476-01215-5 (2004).

This book is a supplementary resource intended to enhance the teaching and learning of a first-year university Calculus course. It can also be used in upper secondary school. It consists of carefully constructed *incorrect* mathematical statements that require students to create counter-examples to disprove them. Some of these statements are the converse of famous theorems, others are created by omitting or changing conditions of the theorems. Some are incorrect definitions and some are seemingly correct statements. Many of the statements are related to common students' misconceptions. In this book the following major topics from a typical single-variable Calculus course are explored: Functions, Limits, Continuity, Differential Calculus and Integral Calculus.

The book can be useful for:

- upper secondary school teachers and university lecturers as a teaching resource
- upper secondary school and first-year university students as a learning resource
- upper secondary school teachers for their professional development in both mathematics and mathematics education

"This book is a welcome and refreshing antidote to the descending spiral of instrumentality. It is offered to those students and those teachers who know that there is more to learning mathematics than completing homework mechanically. It provides the groundwork on which to ascend the spiral of instrumentality towards appreciation and understanding of the mathematics behind the calculus."

Professor John Mason, The Open University, UK

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